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**Technical Report 393**

# UNDERWATER SOUND PROPAGATION-LOSS PROGRAM

## Computation by normal modes for layered oceans and sediments

DF Gordon

17 May 1979

Final Report for Period 1976 — 1978

Prepared for
Naval Sea Systems Command
(NSEA 63R-23) Washington DC 20362

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## ADMINISTRATIVE INFORMATION

The computer program described in this report was developed in the course of work sponsored by Naval Sea Systems Command, Sonar Technology Office (NSEA 63R-23), under Problem SF 52-552-602, NOSC work unit 714-SU10. Some elements of this program have been in development since 1965, but the final modifications, to achieve the current capabilities, and the reporting were done from 1976 to 1978. The computer program is derived from an earlier program developed by MA Pedersen. D White did significant parts of the mathematical analysis. This report was approved for publication 17 May 1979.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A normal-mode program for a sound-speed profile of an arbitrary number of layers has been constructed. It has been used extensively and successfully for 12 years to compute sound propagation in idealized underwater acoustic ducts. Documentation is contained for those who wish to use or modify this program. The FORTRAN statements are given for both the normal-mode and a mode-follower program. The numerical analysis necessary for computing modified Hankel functions of order 1/3 is given. The analysis includes a continued fraction technique.

## OBJECTIVE

Translate well-known differential equation solutions into a working program to compute propagation in underwater acoustic ducts. Document the program methods, to assist users of this and similar programs.

## RESULTS

1. An effective program for computing propagation loss in a layered ocean by normal modes has been developed. Complete documentation for the program is contained herein.

2. Sediment layers are modeled as fluids in which densities, sound speeds, and absorption can be specified. This permits a complete wave solution for bottom reflected sound energy.

3. A continued fraction technique for evaluating asymptotic series is shown to give superior results in evaluating the auxiliary functions required in this program, the modified Hankel functions of order 1/3.

4. A mode follower program given here is useful in tracing eigenvalues. Such traces are needed to understand the eigenvalue structure.

## RECOMMENDATIONS

1. Improve the mode locating ability of this normal-mode program to make it self-contained. It currently requires user interaction to locate eigenvalues.

2. Investigate methods to incorporate the effect of rough boundaries into this program.

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## INTRODUCTION

This report describes a normal-mode program that has been used successfully for 12 years to compute sound propagation in idealized underwater acoustic ducts. The theory and considerations used in developing the program are discussed here, and a copy of the FORTRAN statements are included as appendix A. Appendix B consists of sample inputs and outputs to assist users in gaining familiarity with the program. It is hoped that this report contains sufficient information to allow a user to run the program and to modify it as desired.

This program follows the methods developed by Furry and Freehoffer (ref 1) to compute electromagnetic propagation in the 1940s. Marsh adapted these methods to underwater sound in his doctoral thesis (ref 2). Using this material, Pedersen, at NOSC in the late 1950s, adapted the method to digital computers and developed the programs to compute the auxiliary functions. This original program used two layers to define the sound-speed profile (ref 3). This program was expanded to three layers by DF Gordon and RF Hosmer and finally to the multiple-layer program reported here. In this program the only constraints on the number of layers are computer space and running time. The program is normally configured to permit up to 12 layers.

The earlier programs were used to study sound propagation in ocean surface ducts. Programs that permit more layers have proven useful also for studying propagation in the deep ocean, although the number of modes required generally limits computations to frequencies below 300 Hz. The multiple-layer program has also proven useful in modeling sediment layers and thus in computing shallow-water propagation.

The principal limitation in the application of this program to real-world situations is the requirement of ideal conditions: boundaries must be smooth and horizontal, and no variation of boundary conditions with range is permitted. Despite this limitation, the program has proven useful in predicting and explaining acoustic propagation and has applications in a number of related areas. These include checking other types of wave-theory models or corrections such as caustic corrections; determining group velocities, dispersion curves, and reflection coefficients; and determining acoustic coupling between ducts.

The following paragraphs describe the specific topics covered by the sections in this report. In GENERAL SOLUTION are the equations required to solve the wave equation with the boundary conditions used here. DETERMINANT is part of the basic solution but is concerned with the particular numerical method used in this program to evaluate the conditions imposed by the boundaries. Other approaches could be used instead. A later section, NUMERICAL BREAKDOWN, is also part of the basic solution, but deals with special numerical problems that have arisen but are not apparent from the basic equations.

---

1. The Bilinear Modified-Index Profile, by WH Furry, in Propagation of Short Radio Waves, DE Kerr, ed; MIT Rad Lab series, vol 13, p 140-168, McGraw-Hill, New York, 1951.
2. Navy Underwater Sound Laboratory Report 111, Theory of the Anomalous Propagation of Acoustic Waves in the Ocean, by HW Marsh, 1950.
3. Normal-Mode Theory Applied to Short-Range Propagation in an Underwater Acoustic Surface Duct, by MA Pedersen and DF Gordon; J Acoust Soc Am, vol 37, p 105-118, January 1965.

FINDING EIGENVALUES deals with the philosophy of eigenvalue location employed by this program, which essentially leaves this function to the user, the program only serving as a tool. It shows how the program is used to make computations.

Several "automatic" mode finding versions of this program have been developed to the point of accommodating certain classes of profiles. However, they need further development and have not yet been reported.

SOUND SPEED PROFILE indicates the required equations for curve fitting and the various ways the sound speed can be read in on cards. A continuous water profile can be entered quite simply, but sediment layers with sound speed discontinuities and absorption gradients can become complicated.

REFLECTION COEFFICIENTS AND OTHER AUXILIARY OUTPUTS describes a short subroutine that computes reflection coefficients for any mode at a given profile interface. Intermode interference lengths and mode damping coefficients are also discussed.

COMPUTATION OF THE MODIFIED HANKEL FUNCTIONS gives the analysis necessary for computing these functions. The use of continued fractions to evaluate an asymptotic series is discussed. To facilitate running the program on computers of different word length, this section provides the information required to optimize the functions for the different word lengths.

MODE FOLLOWER PROGRAM describes a separate but related program for investigating the eigenvalues themselves rather than using them to compute propagation losses.

## GENERAL SOLUTION

The derivation of the normal-mode solution has been discussed from various points of view (eg ref 1, 4, 5). Only an outline is given here. In general, the time-independent wave equation is written in polar coordinates and the azimuthal coordinate is dropped under the assumption that the field is independent of azimuthal direction. Thus

$$(1/r)\,(\partial/\partial r)\,[r(\partial\psi/\partial r)] + (\partial^2\psi/\partial z^2) + (\omega^2/c^2)\,\psi = 0, \qquad (1)$$

where $\psi$ is the velocity potential, c the sound speed, and the independent variables are depth, z, and range, r.

Equation (1) is then separated into range- and depth-dependent parts with a separation constant $\lambda$. The separation is possible when the sound speed is a function of depth only. After accounting for the source discontinuity and the outgoing radiation condition, integrating over all real values of the separation constant, and normalizing, one can find the solution for a field point in terms of propagation loss H as follows:

---

4. Naval Air Development Center Report NADC-72002-AE, Normal Mode Solutions and Computer Programs for Underwater Sound Propagation, by CL Bartberger and LL Ackler, 4 April 1973.
5. A Normal Mode Theory of an Underwater Acoustic Duct by Means of Green's Functions, by RL Deavenport; Radio Sci, vol 1, p 709–724, 1966.

$$H = -10 \log \left| \rho_s \rho_h \pi \sum_{n=1}^{N} H_0^2 (\lambda_n r)\, U_n(z)\, U_n(z_0) \right|^2 + \alpha_A r, \tag{2}$$

where r is the range, $z_0$ is the source depth, z is the receiver depth, $H_0^2$ is the Hankel function of order zero, second type, $\lambda_n$ is the *nth* eigenvalue, $U_n$ is the depth function for mode n, and $\rho_s$ and $\rho_h$ are the densities at source and receiver. The sum is over the number of modes, N, making a significant contribution. The final term contains the volume attenuation coefficient, $\alpha_A$. From Thorp (ref 6), $\alpha_A$ in dB/m is computed by the relationship

$$0.9144\, \alpha_A = 0.0001\, F^2/(1 + F^2) + 0.04\, F^2/(4100 + F^2), \tag{3}$$

where F is the frequency in kHz. Improved equations or those for specific ocean areas can be easily substituted. The depth function, $U_n$, is a solution to the depth-dependent part of the separated wave equation

$$d^2U/dz^2 + [\omega^2/c^2(z) - \lambda^2]\, U = 0, \tag{4}$$

where

$$\omega = 2\pi f$$

and f is the frequency, in Hz.

A closed-form solution to eq (4) can be obtained when the reciprocal sound speed squared or squared index of refraction is a linear function of depth. That form is used in this program, and sound speed in each layer is expressed as follows:

$$[c_i/c(z)]^2 = 1 - 2\, \gamma_i\, (z - z_i)/c_i, \tag{5}$$

where $c_i$, $z_i$, and $\gamma_i$ are the sound speed, depth, and sound-speed gradient, respectively, at the top of layer i. Up to 12 such layers are permitted by the program, for modeling the sound-speed profile.

With this expression for sound speed, solutions to eq (4) can be expressed in terms of solutions to Stokes' equation

$$h'' + zh = 0. \tag{6}$$

Only a simple change in independent variable is required from z to $\zeta$, where

$$\zeta_i(z) = \left[ a_i^3 (z - z_i) + \omega^2/c_i^2 - \lambda^2 \right] / a_i^2 \tag{7}$$

and

$$a_i^3 = -2\gamma_i\, \omega^2/c_i^3. \tag{8}$$

---

6. Analytic Description of the Low-Frequency Attenuation Coefficient, by WH Thorp; J Acoust Soc Am, vol 42, p 270, 1967.

The solutions to Stokes' equation that are used are the modified Hankel functions of order 1/3, $h_1(\zeta)$ and $h_2(\zeta)$. The depth function is a linear combination of these two independent solutions:

$$F_{n,i}(z) = A_{n,i}h_1(\zeta_n) + B_{n,i}h_2(\zeta_n), \tag{9}$$

where $F_n$ is the unnormalized form of $U_n$. The coefficients $a_{n,i}$ and $B_{n,i}$ for mode n in layer i are determined to satisfy boundary conditions, which will be listed below. Values of $\lambda_n$ for which the boundary conditions can be satisfied are the eigenvalues.

The first boundary condition is the radiation condition. It is satisfied by using a negative sound-speed gradient in the deepest layer, which extends to infinite depth, and by letting the depth function there be proportional to $h_2$ only. That is,

$$F_n(z) = B_n h_2(\zeta_n). \tag{10}$$

At the surface the depth function is zero:

$$F_n(0) = 0, \tag{11}$$

and at layer interfaces, $\rho U$ and its depth derivative are continuous:

$$\rho_i F_{n,i}(z) = \rho_{i+1} F_{n,i+1}(z); \tag{12}$$

$$dF_{n,i}(z)/dz = dF_{n,i+1}(z)/dz. \tag{13}$$

Here $\rho_i$ is the density in layer i, and the excess acoustic pressure, p, is given by

$$p = \rho U.$$

If U is assumed to be the vertical component of the velocity potential, eq (12) and (13) are equivalent to requiring that the pressure and the vertical component of particle velocity be continuous across the layer interface.

Applying these boundary conditions to a sound-speed profile consisting of M layers results in 2M – 1 linear equations in $h_1$ and $h_2$. They are homogeneous in that the constant is zero in each equation. There are M–1 coefficients $A_i$ to be determined and M coefficients $B_i$. These coefficients can therefore be determined within a constant of proportionality D, provided the system of equations is linearly dependent. That is, the 2M – 1 square matrix of coefficients of $A_i$ and $B_i$ must be of rank 2M – 2 or less. Its determinant will then be zero. This is the eigenvalue condition. Values of $\lambda$ must be found which make the determinant zero. This determinant, G, is discussed in more detail in a later section.

Zeroes of the determinant, G, are found by using the secant method. The variable in this iterative method can as well be some function of $\lambda$ as $\lambda$ itself, and we use the following complex phase velocity (v):

$$\lambda_n = \omega/v_n. \tag{14}$$

To find a v that is a root of G requires an initial guess, $v_1$, where the subscript 1 refers to the step in the iteration and a small increment, $\delta_1$. Each succeeding estimate is given by the relationship

$$v_{j+1} = v_j + \delta_j,$$

where

$$\delta_j = -(v_j - v_{j-1})\, G_j/(G_j - G_{j-1}). \tag{15}$$

The details of this iterative process are given in a later section.

When an eigenvalue $v_n$ is found, the coefficients are then evaluated. One coefficient can be given an arbitrary value, so $A_1$ is set to $\rho_1 h_2[\zeta_1(0)]$. From eq (11), $B_1$ is then $-\rho_1 h_1[\zeta_1(0)]$. Pairs of equations (eg (12) and (13)) for each successive interface can then be used to evaluate the next $A_i$ and $B_i$ as discussed later.

Finally the normalizing factor, $D_n$, for mode n is obtained by the relationship

$$D_n = \int_0^\infty \rho F_n^2(\zeta)\, dz. \tag{16}$$

This equation follows from the orthogonality of the depth functions. It is not the pressure, however, which is proportional to $\rho U$, but $\rho^{1/2} U$ that is orthogonal (ref 7). Therefore, $D_n$ must be determined such that the integral of $\rho U^2$ is 1.

From Stokes' equation (eq (6)) and eq (7–9), the integral of $F^2$ takes the form

$$\int_{z_i}^{z_{i+1}} F^2(\zeta)\, dz = \left[\zeta_i(z)\, F^2(\zeta)\Big/a_i + F'^2(\zeta)\Big/a_i^3\right]\Bigg|_{z_i}^{z_{i+1}}. \tag{17}$$

Therefore

$$D_n = -\rho_1^3 W^2/a_1 + \sum_{i=1}^{n-1} \left\{ \rho_i[\zeta_i(z_{i+1})/a_i - \rho_i \zeta_{i+1}(z_{i+1})/(a_{i+1}\rho_{i+1})]\, F_i^2(z_{i+1}) + \left(\rho_i/a_i^3 - \rho_{i+1}/a_{i+1}^3\right) F_i'^2(z_{i+1}) \right\}, \tag{18}$$

where eq (12) and (13) have been used to combine terms at each interface. The derivative of F takes the form

$$F_i'(z_{i+1}) = a_i \left\{ A_i h_1'[\zeta_i(z_{i+1})] + B_i h_2'[\zeta_i(z_{i+1})] \right\}. \tag{19}$$

The Wronskian, W, is an imaginary constant (see eq (85)) and is the contribution of eq (17) at the surface:

$$W = -1.45749544104i.$$

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7. Some Effects of Velocity Structure on Low-Frequency Propagation in Shallow Water, by AO Williams; J Acoust Soc Am, vol 32, p 363-365, March 1960.

The depth functions are normalized by the relationship

$$U_n(z_0)\, U_n(z) = F_n(z_0)\, F_n(z)/D_n. \tag{20}$$

The functions F and F′ used in computing $D_n$ are conveniently assembled from the elements of the determinant and the coefficients $A_i$ and $B_i$. This requires care in developing the computer code, because F is always multiplied by $\rho$ and F′ has the term $a_i$ in it. The surface differs from the other layers in that $F_1$ is zero there and $F'_1$, by eq (19), is $a_1 W$. However, because $\rho_1$ appears as a factor in the coefficients of $F_1$, the actual value of $F'_1$ at the surface in the computation is $\rho_1\, a_1 W$. This factor of $\rho_1$ together with the $\rho_1^{1/2}$ needed for orthogonality, when squared, gives the $\rho_1^3$ of eq (18).

## DETERMINANT

Normal modes are determined by finding the eigenvalues of a characteristic equation which, in turn, is obtained by setting a determinant to zero. The determinant is obtained from the coefficient matrix of a set of linear, homogeneous equations expressing the boundary conditions as given by eq (10) – (13). Since the method of handling this determinant is a central feature of this normal-mode program, it is given in detail here.

The first line of the matrix is taken from eq (11) as

$$B_1 \rho_1 h_2 [\zeta_1(0)] + A_1 \rho_1 h_1 [\zeta_1(0)] = 0. \tag{21}$$

At each profile interface, i, where i numbers the interfaces below the surface from 1 to N–1, the two boundary conditions given by eq (12) and (13) are

$$B_i \rho_i h_2 [\zeta_i(z_{i+1})] + A_i \rho_i h_1 [\zeta_i(z_{i+1})] - B_{i+1} \rho_{i+1} h_2 [\zeta_{i+1}(z_{i+1})] - A_{i+1} \rho_{i+1} h_1 [\zeta_{i+1}(z_{i+1})] = 0 \tag{22}$$

and

$$B_i a_i h'_2 [\zeta_i(z_{i+1})] + A_i a_i h'_1 [\zeta_i(z_{i+1})] - B_{i+1} a_{i+1} h'_2 [\zeta_{i+1}(z_{i+1})] - A_{i+1} a_{i+1} h'_1 [\zeta_{i+1}(z_{i+1})] = 0 . \tag{23}$$

The coefficients of $A_i$ in the first equation and $B_{i+1}$ in the second will be the diagonal elements of the matrix. The nonzero elements of the matrix will therefore be no more than two places from the diagonal. The matrix can be stored in the computer in an array of size (2M–1) × 4, where M is the maximum number of layers in the sound-speed profile. In the final layer, $A_N h_1$ is omitted, as in eq (10). In the program, the real and imaginary parts are stored in separate arrays.

The sparseness of the matrix permits efficient evaluation by a triangularization process of row reduction. For each pair of rows representing a pair of equations given by eq (22) and (23), the first element from the first equation and the first two from the second equation must be set to zero by subtracting the proper multiple of preceding rows. The determinant is then the product of the diagonal elements of the triangularized matrix. The value of the determinant, G, is used in eq (15) to find the roots by iteration.

Note that a value of v that makes this determinant zero, or near zero, ordinarily is zero because only one diagonal element is very small. For trapped modes this element is at the row representing the first interface below the mode, ie the interface just below the layer of positive gradient in which the sound speed is equal to the mode phase velocity. For unstrapped modes it is usually the final diagonal element that is small. Thus the layers in which the sound speed is greater than the phase velocity of a mode do not greatly affect the eigenvalue. Eigenvalues are determined mainly by those parts of the sound-speed profile that are less than the phase velocity.

When an eigenvalue is found, the coefficients $A_i$ and $B_i$ must next be evaluated. As mentioned earlier, one coefficient can be arbitrarily chosen. This is done, and eq (21) is satisfied by letting

$$A_1 = \rho_1 h_2[\zeta_1(0)]$$

and

$$B_1 = -\rho_1 h_1[\zeta_1(0)]. \tag{24}$$

The factor $\rho_1$ is used simply because the number containing it is easily available in the program. It is divided out by the normalizing factor, D. Eq (22) and (23) can then be used to evaluate the remaining coefficients, but the triangularized form of the matrix yields the coefficients with less computation. If $g_{ij}$ is the element in the *ith* row and *jth* column of the triangularized matrix, then by Cramer's rule,

$$B_i = A_{i-1}\, g_{2i-2,2i-2}\, g_{2i-1,2i}/E_i$$

and

$$A_i = -A_{i-1}\, g_{2i-2,2i-2}\, g_{2i-1,2i-1}/E_i,$$

where

$$E_i = g_{2i-2,2i-1}\, g_{2i-1,2i} - g_{2i-2,2i}\, g_{2i-1,2i-1}. \tag{25}$$

A simpler form is used for $B_N$ in the final layer since there is no $A_N$ there.

In certain situations numerical problems can arise in evaluating the determinant. These require some extra tests in the subroutine that makes the evaluation. The extra tests will be discussed in the section, NUMERICAL BREAKDOWN. A more routine problem is the loss of accuracy that can arise in subtractions in the row reduction of the matrix. This loss results in less sharpness of convergence to a root. The size of the determinant, G, can be 14 orders of magnitude less at a root than at the general background near the root. This variation occurs because the modified Hankel functions can be computed to about 14-place accuracy in a computer with 18 decimal places available. Modes usually converge to 10 or 12 places; thus a few places are lost in evaluating the determinant. In some profiles, usually those with multiple ducts or those in which propagation through bottom sediments plays a large part, the convergence can be much poorer. Modes need to converge to about 4 places to be reliable for computing losses, and convergence occasionally fails to meet this requirement. The only current cure for this loss in accuracy is to go to higher-precision arithmetic or to compute the modified Hankel functions to greater accuracy. For instance, a standard

matrix triangularization routine that uses full row and column pivoting has been tried with no resultant increase in accuracy.

## FINDING EIGENVALUES

There are versions of this program under development that will locate the eigenvalues and do the entire computation without user intervention. Currently, however, these versions are reliable only for the simpler types of profiles – usually those with only one duct – and are not ready to be reported. Locating eigenvalues with the standard version of the program is discussed here.

The standard version of the program requires the user to find the eigenvalues. In this version, each time an eigenvalue is determined by iteration, the resulting value is stored and counted as an eigenvalue. Therefore, the user must ensure that all iterations result in good roots, that all required modes have been determined, and that no modes are present more than once. In most cases the user must expect to make more than one computer run to obtain this result.

### CONTROL CARDS

The user controls the eigenvalue determination by using any of four different types of control cards. The first type specifies an initial value for v and an initial step size, $\Delta v$. These are both complex numbers with a real and an imaginary part. G is then evaluated at v and at $v + \Delta v$ to start the iteration. These are essentially the $v_j$ and $v_{j+1}$ of eq (15). If these two trial eigenvalues are in the vicinity of a root, the iteration will converge to that root.

The second type of card specifies a line segment in the complex plane, along which a search for eigenvalues, v, is made. The end points of the line are given along with the number of equally spaced points at which the line is to be divided. G is then evaluated at each successive division point along the line until a relative minimum in $|G^2|$ is found, indicating that a root is nearby. The iterative process is applied to find the root. The initial step size, $\Delta v$, is first computed to bring the second evaluation at $v + \Delta v$ as close as possible to the true root. This is done by using the point which resulted in minimum $|G^2|$ and the points on either side of it to determine the minimum of the parabola passing through them. If $v - h$, v, and $v + h$ are the three points at which G was evaluated, it follows that the distance from v to the minimum of the parabola

$$\Delta v = h[G(v + h) - G(v - h)]/2[2G(v) - G(v + h) - G(v - h)]. \qquad (26)$$

When the iteration is complete, the eigenvalue is recorded and the program continues to step along in the direction of the given line, checking again for a minimum. However, the stepping is resumed from the newly located root rather than from the approximate location where the minimum was detected. With this correction in position, the designated line does not have to hug the curve on which the eigenvalues are located because it is corrected at each eigenvalue.

This method of finding eigenvalues has proven very successful. Its main utility arises, though, because the eigenvalues of the trapped modes have negligible imaginary parts and the

search can be made along the real line. In simple profiles this can often give a successful set of modes on the first try. Usually, only the three initial eigenvalues need to be located by this means because further eigenvalues can be located by extrapolation on the previous three. This is the function of the third type of control card.

The third type of card specifies the number of additional modes to be determined by extrapolation. The starting value of each eigenvalue is determined by extrapolating from the three most recently determined eigenvalues to find v. The step size, $\Delta v$, is chosen as 0.0001 times the distance between the last two eigenvalues. The exact eigenvalue is then determined by iteration. The extrapolation is the simple parabolic form for equal steps:

$$v = 3v_n - 3v_{n-1} + v_{n-2}. \tag{27}$$

This method of locating modes works well when the modes lie along a smooth curve, as usually occurs for single ducts. But this relationship does not always occur for profiles with multiple ducts.

The final control card is punched by the program when requested and contains the correct eigenvalue to full precision. Upon encountering this card, the program does not iterate, but instead evaluates G for this eigenvalue and stores this value of G as the next eigenvalue. A deck of such completed eigenvalues can be stored, saving the expense of recomputing the eigenvalues for a given profile and frequency.

### ITERATION TERMINATION

A full description of the iteration of eq (15) should include the method of termination. The usual criterion for stopping is that G fails to become smaller. As G approaches minimum size, however, round-off error can act as noise so that G is no longer a predictable function of v. The denominator of eq (15) can then be very small by chance, resulting in a large value for $\delta_i$. If this happens, the next value of v, which was as near to the root as possible, will be far away. A much better convergence criterion is that $\delta_i$ has reached a minimum in absolute value. In the program, iteration is stopped when $|\delta 2|$ exceeds the previous value by a factor of 2. However, this criterion is not applied until three iterative steps have been completed, to permit the process to become well established. An upper limit of 15 iterative steps is permitted. We have not found an improvement on the root after 15 steps.

## SOUND SPEED PROFILE

The normal mode program requires as inputs the depth of each layer and the sound speed and sound speed gradient at the top of each layer. These variables are mapped into the dimensionless internal variables of the program by eq (7). The purpose of the sound speed profile processing portion of the program is to accept the profile parameters in a form convenient for the user and to translate them into the required sound speeds and gradients.

The first function of the processing program is to make the sound speed continuous at interfaces. This is done simply by using the sound speed at the bottom of one layer as the sound speed at the top of the next. It may be necessary to compute the sound speed at the bottom of the layer. The necessary parameters will have been given. Occasionally a

discontinuity in sound speed is required, as when modeling an interface between water and sediment. The user indicates this by specifying the sound speed at the top of the layer. If left blank, the program provides the sound speed necessary for continuity.

A second function of the processing program is to permit a layer to be defined by the sound speed at top and bottom of the layer rather than by one sound speed and one gradient. Note that the profile form as given by eq (5) is a two-parameter curve.

The last layer extends to infinite depth, so a gradient must be specified at the top of it. However, this gradient can be specified by giving a depth and sound speed point below the last layer. The program handles this by checking to see if the gradient of the last given layer is unspecified. If it is, the number of layers is reduced by one, which causes the last layer to be only the required extra point determining the final gradient. This final gradient must always be negative, as is required by the boundary conditions. The program user must ensure that this gradient is negative and that no gradient is zero. A zero gradient will appear in the denominator of eq (7).

These functions of the profile processing program are relatively simple, but an additional capability used to model sediment bottoms greatly increases the complexity of the program. The capability required is to specify the absorption in a layer by adding an imaginary part to the sound speed. In older versions of this normal mode program an imaginary part, expressed as an absorption coefficient, could be added to the sound speed at the top of the layer. This imaginary part is small compared to the real part. Since the gradient was assumed real at the top of the layer, the imaginary part was initially not changing with depth and it usually changed only a minor amount through the depth of the layer. However, this small change could not always be relied upon. Also Hamilton (ref 8) has published data on absorption gradients in sediment layers, so more precise control of this part of the sound speed function is needed to model sediment layers. Therefore, a more comprehensive profile processing routine has been incorporated in the normal mode program. This curve-fitting process is described below.

The following quantities can be input for each layer depth starting at the surface:

Depth of top of the layer

Sound speed at top of layer

Sound speed at bottom of layer

Real part of sound speed gradient at top of layer

Attenuation in loss per km at the top of the layer

A similar attenuation at the bottom of the layer

Density in the layer

The density is a constant in the layer and as such requires no further curve fitting. Redundant parameters are left blank on input cards. In some cases negative values serve as flags to indicate specific treatment. For instance a negative value of absorption at the top of a layer

8. Sound Attenuation as a Function of Depth in the Sea Floor, by EL Hamilton; J Acoust Soc Am, vol 59, p 528–535, March 1976.

directs the program to use the same imaginary part of sound speed as occurred at the bottom of the previous layer. Similar flags at the bottom of a layer are discussed later.

Absorption per Hz is given in units of decibels per km (or kiloyard). The quotient of absorption over frequency is used because Hamilton (ref 8) usually considers absorption (or attenuation) as proportional to frequency with a coefficient k. We use the symbol h instead. That is,

$$\alpha = hf.$$

We interpret $\alpha$ to be in units of dB per km and f in Hz, whereas Hamilton uses dB per m and kHz; but the coefficients h and k remain equal.

The complex wave number in layer i is represented as

$$k_i = \omega/C_i$$

$$= \omega \mathrm{Re}C|C|^{-2} - i\omega \mathrm{Im}C|C|^{-2}. \tag{28}$$

A plane wave will be attenuated $\alpha$ dB per km if

$$\mathrm{Im}k_i = -\alpha/(20\,000 \log e)$$

$$= -\pi A f, \tag{29}$$

where

$$A = h/(20\,000\,\pi \log e).$$

By equating the imaginary part of $k_i$ in eq (28) and (29), the imaginary part of $C_i$ is found to be as follows:

$$\mathrm{Im}C_i = 1/A - [1/A^2 - (\mathrm{Re}C_i)^2]^{1/2}. \tag{30}$$

If $\alpha$ is zero, which is the case usually used in water layers, eq (30) cannot be used; but the imaginary part of C is then simply zero. These two cases are treated separately in the program.

When sound speed is given at the top and bottom of layer i, the imaginary parts of the sound speeds are determined by eq (30) and the only curve fitting task is to determine the gradient $\gamma_i$. Solving eq (5) for $\gamma_i$,

$$\gamma_i = C_i(C_{i+1}^2 - C_i^2)/2C_{i+1}^2(z_{i+1} - z_i). \tag{31}$$

The gradient is a complex number since the C's here are complex. The z's are real.

A second version of this computation arises if the gradient is required to be a real number. In this case, which is used to match older versions of the program, an additional parameter must be left unspecified and this parameter is Im $C_{i+1}$. This is equivalent to having the sound absorption at the bottom of the layer unspecified. Therefore, a negative number input for this parameter is used as a flag to call for this particular fitting procedure.

For this situation, given Re $C_i$, Im $C_i$, Re $C_{i+1}$, and making $\gamma_i$ real, the determination of $\gamma_i$ and Im $C_{i+1}$ is not simple. When $\gamma_i$ is eliminated from the real and imaginary parts of eq (31), a quartic equation in Im $C_{i+1}$ results. Rather than derive an algebraic solution to this equation, it is solved by iteration under Newton's method. A good first guess at the solution is Im $C_{i+1} \cong$ Im $C_i$. Four iterations usually give an accurate root. The equation is

$$\text{Im } C_i\,(\text{Im } C_{i+1})^4 + \left[\text{Im}(C_i)^3 + 2(\text{Re } C_{i+1})^2\,\text{Im } C_i\right](\text{Im } C_{i+1})^2$$
$$+ 2\text{Re } C_{i+1}\,\text{Re}(C_i)^3\,\text{Im } C_{i+1}$$
$$+ \text{Im } C_i(\text{Re}C_{i+1})^4 - (\text{Re } C_{i+1})^2\,\text{Im}(C_i^3) = f(\text{Im } C_{i+1}). \qquad (32)$$

The root is then found:

$$(\text{Im } C_{i+1})_j = (\text{Im } C_{i+1})_{j-1} - f/f'.$$

The gradient, $\gamma$, is next given by the relationship

$$\gamma_i = \left\{\text{Im } C_i\left[(\text{Re } C_{i+1})^2 - (\text{Im } C_{i+1})^2\right] + 2\text{Re } C_i\,\text{Re } C_{i+1}\,\text{Im } C_{i+1} - \text{Im}(C_i^3)\right\} \Big/$$
$$[4\,\text{Re } C_{i+1}\,\text{Im } C_{i+1}\,(z_{i+1} - z_i)]. \qquad (33)$$

Because the root of eq (32) may not be exact, Im $\gamma_i$ may not be exactly zero. This slight error can be transferred to $C_{i+1}$ by using the computed real $\gamma_i$ to recompute $C_{i+1}$. This is done in the program by transferring to a portion of the program already designed to do this.

When sound speed and gradient at the top of the layer are given, the parameters required by the program are all given. The sound speed at the bottom of the layer is routinely computed, however, because it may be required to make the next layer continuous. Equation (5) is used to determine the sound speed at depth $z_{i+1}$, which is the depth of the bottom of the layer. This is straightforward, but several complications arise. Only the real part of the gradient at the top of the layer is used as an input because situations have not arisen that require that the imaginary part of the gradient be specified. Often the attenuation is given at both top and bottom of the layer. That is, Re $C_i$, Im $C_i$ and Re $\gamma_i$ are given, plus a relationship between Re $C_{i+1}$ and Im $C_{i+1}$. The imaginary part of the gradient, Im $\gamma_i$, must be determined as well as both real and imaginary parts of the sound speed at the layer bottom. The derivation of this case is not trivial.

One relationship between the real and imaginary parts of the sound speed is given by eq (28) and (29). From these equations at $C_{i+1}$ we derive

$$A(T - i) = 2/C_{i+1}, \qquad (34)$$

where

$$T = \text{Re } C_{i+1}/\text{Im } C_{i+1}.$$

Substituting this expression for $C_{i+1}$ into eq (31) and equating real parts gives a quadratic expression for T which has a usable root of

$$\mathrm{Re}(C_i^3)T = -\mathrm{Im}(C_i^3) - \left\{ [\mathrm{Im}(C_i^3)]^2 + \mathrm{Re}(C_i^3)B \right\}^{1/2}, \tag{35}$$

where

$$B = \mathrm{Re}(C_i^3) - 8\ \mathrm{Re}\ \gamma_i(z_{i+1} - z_i)/A^2 + 4\ \mathrm{Re}\ C_i/A^2.$$

From eq (34),

$$\mathrm{Re}\ C_{i+1} = 2T/A(T^2 + 1)$$

and (36)

$$\mathrm{Im}\ C_{i+1} = RC_{i+1}/T.$$

The gradient can now be evaluated by eq (31) to find its imaginary part.

Equations (34) and (35) cannot be used if the attenuation at the bottom of the layer is given as zero. Therefore an alternate form must be used. This form is much simpler than the previous case, since $C_{i+1}$ is real.

$$C_{i+1} = \left\{ \mathrm{Re}(C_i^3) \Big/ [\mathrm{Re}\ C_i - 2\ \mathrm{Re}\ \gamma_i\ (z_{i+1} - z_i)] \right\}^{1/2} \tag{37}$$

$$\mathrm{Im}\ \gamma_i = [\mathrm{Im}\ C_i - \mathrm{Im}(C_i^3)/C_{i+1}^2]\ [2(z_{i+1} - z_i)]^{-1} \tag{38}$$

Finally, if the special case, $\gamma_i$ real, is specified by inputting a negative value for absorption, eq (31) can be used directly to give

$$C_{i+1}^2 = C_i^3/[C_i - 2\gamma_i(z_{i+1} - z_i)]. \tag{39}$$

To evaluate the square root, let

$$C_{i+1}^2 = a + bi.$$

Then

$$\mathrm{Re}\ C_{i+1} = \left\{ \left[ a + (a^2 + b^2)^{1/2} \right] \Big/ 2 \right\}^{1/2} \tag{40}$$

and

$$\mathrm{Im}\ C_{i+1} = b/2\ \mathrm{Re}\ C_{i+1}. \tag{41}$$

## NUMERICAL BREAKDOWN

A situation arises frequently in which a very small depth function must be computed from the difference of two large numbers. A wrong answer results if this accuracy loss exceeds the word size of the computer. The best way that has been found to avoid this is to check for it within the program and arbitrarily replace the wrong number. In checking for this, a constant, called T-lim in the computer program, is compared to the argument of the

modified Hankel functions or to the argument of the exponential function within modified Hankel functions. A T-lim value of 25.0 is used in the program, but a smaller number occasionally is required. The program user can alter T-lim by appropriate input cards (Key 8 = 1 followed by a new value of T-lim). The next few paragraphs demonstrate the symptoms of this problem, so as to assist a user in recognizing the problem. The remainder of this section describes the modifications that have been made to the computer program to correct this loss of accuracy.

The solid line of figure 1 shows a simple surface duct and the phase velocities of the first three modes at 3 kHz. For this profile, the depth function of mode 1 is shown in figure 2. The solid line is the depth function as computed by a program that does not correct for numerical breakdown. The dashed line shows the correct depth function below a depth of 71 m. This result was determined from Airy functions, not from the program. Between depths of 71 to 100 m, the program cannot compute the depth function accurately. In the second layer, which starts at a depth of 100 m, the function can be computed accurately but it is incorrectly placed by the boundary condition that requires the depth functions to be continuous at interfaces. The slope of the depth function was correctly computed as indicated by the identical shape of the three depth functions in the second layer. The shape is such as to make the correct depth function continuous in slope across the interface.

The breakdown in accuracy at a depth of 71 m occurred when $\zeta$ had a value of -8.4. ($\zeta$ is given by eq (7) and is the argument of the modified Hankel functions.) A negative value of $\zeta$ occurs when the mode phase velocity is less than the speed of sound. Since the ray of the same phase velocity cannot reach such a region, the sound field there is a diffracted field. The mode depth function is therefore small at such depths. In the figure, the depth function amplitude at the breakdown point is about 7 orders of magnitude (or in terms of propagation loss, 140 dB) down from its maximum. Equations (62), (66), and (68), which will be given for the modified Hankel functions, indicate that the argument of the exponential term is $2/3(8.4)^{3/2}$, or 16.2. The functions $h_1$ and $h_2$ will thus be about $10^7$ in magnitude at a depth of 71 m. These large values and their small difference account for the approximate accuracy loss of 14 decimal places, which is the general accuracy of the modified Hankel functions.

Incorrect behavior in the depth function usually occurs when $\zeta$ is about -8.4. In some more complicated profiles, however, where accuracy is lost in row reduction of the determinant, the depth functions may become incorrect at values of $\zeta$ that are less in absolute value. When this problem occurs it can be diagnosed by plotting the depth function of the mode and noting the steep positive slope through some depth interval as in figure 2. When that occurs, the value of T-lim should be decreased.

Incorrect depth functions can cause errors in propagation loss computations in two ways. In figure 2, the solid-line depth function, because of its large size, can cause losses to be too low at a depth of around 100 m. The second error would occur if the duct were deeper, say 110 m. At this depth the erroneous segment of depth function in figure 2 would reach a value of about $10^{-1}$, where it would be larger than the correct lobe of the depth function near the surface. With this extra area under the curve, the normalizing factor would be increased significantly and would reduce the size of this entire depth function. Thus, losses near the surface would be larger because of the loss in size of mode 1.

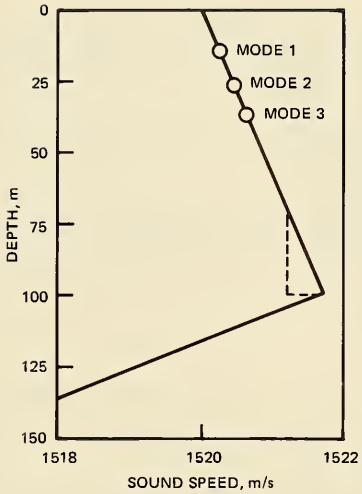


Figure 1. A two-layer sound speed profile for a surface duct. The phase velocities of the first three modes at 3 kHz are marked. The broken line shows a modification of the upper layer to prevent numerical breakdown in mode 1.

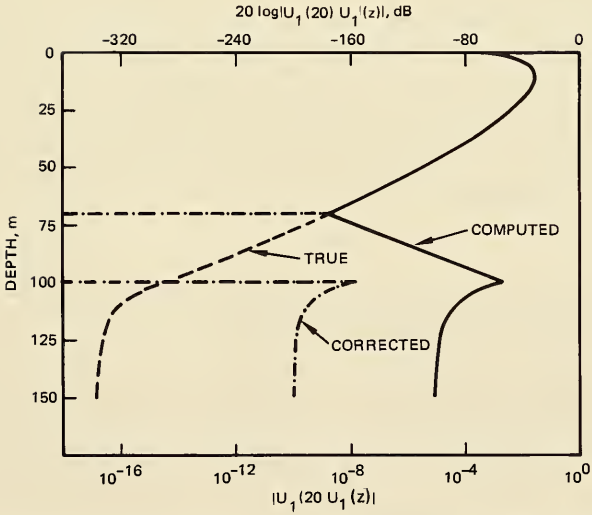


Figure 2. Depth function of mode 1 at 3 kHz, showing error in the computed function. The true function cannot be computed without increasing computer word length, but the corrected value can and it will not cause a large error in the mode sum.

The standard correction to mode 1 is shown in figure 2 by the dot-dashed line. In the depth interval where $\zeta < -8.4$, the function is set to zero. The values of the depth function at greater depths result from a modification in the values used in the determinant.

The corrected values in figure 2 are not equal to the true value of the depth function, but they are small enough that they do not alter the propagation loss to a tenth of a decibel when a full set of modes is used. When the source or receiver is at a depth where such corrections are necessary, the mode can be omitted from the computation. Thus, properly omitting modes would solve the above problems except for the cases where the normalizing factor, D, is affected. In these cases, losses cannot be computed accurately without the corrections.

## PROGRAM MODIFICATIONS

The modification is approximately equivalent to modifying the sound speed profile as shown by the broken line in figure 1. In effect, the sound speed is not allowed to become enough greater than the phase velocity of the mode being considered to cause problems.

The limitation on $\zeta$ is accomplished at three different places in the normal mode program. It is not clear that this is the best way to handle the problem and it may be redundant, but it appears to be an adequate solution. These three corrections will be described next. Finally a correction to the determinant program is described which is necessary because the limiting of $\zeta$ can cause false zeroes in the determinant.

In the subroutine SETUP the elements of the determinant are computed by determining $\zeta$ at the top and bottom of each layer and then calling the modified Hankel function program. At the top of each layer, Re $\zeta$ is set to -7.5 if its value was less. However, this is done in an iterative loop in which the real part of $\omega/C_i$ in eq (7) takes on a sufficiently larger value while its imaginary part is fixed. This is done to retain the absorptive properties of a layer when its sound speed is in effect being reduced. It has been found unnecessary to make the above constant, -7.5, a function of T-lim which the user can vary, because an oversized value at the top of a layer is not as critical as at the bottom. At the bottom of a layer, several tests are made. If the real part of $\zeta$ has decreased past the limit at some depth between the top and bottom of the layer, it is set at the limit. This limit, called S-lim in the program, is related to T-lim by the relationship

$$S = -(T)^{2/3} \tag{42}$$

where S and T are the two limits. If Re $\zeta$ is less than -7.5 throughout the layer, it is simply set at -7.5. Such a layer has negligible effect on a mode.

In program MAIN at the location where depth functions are computed for given depths, a process similar to that above is used. To evaluate the depth function in a given layer, $\zeta$ is first evaluated at the top of the layer. The real part of $\zeta$ is then limited as in the program SETUP above. Next $\zeta$ is evaluated at the given depth by adding the depth-dependent part onto the value at the top of the layer which may be the modified value. If this final value is less than S-lim, the depth function is set to zero. If it is greater than S-lim, the function is computed in the usual way.

The imaginary part of $\zeta$ can be large if the eigenvalue has a large imaginary part or if the speed of sound in the layer has a large imaginary part. When this happens the imaginary part of $2/3\ \zeta^{3/2}$, which appears as an exponential in the modified Hankel functions, may become large in absolute value even though Re $\zeta$ has been limited. A final check is therefore made before the exponential is computed. If Im $\zeta^{3/2}$ is greater than T-lim, $\zeta$ is reduced in amplitude to the size at which it will equal T-lim. The angle of $\zeta$ in the complex plane is preserved.

This limitation of the exponential can be viewed in another way. In a following section the two components of the modified Hankel functions, $F_1$ and $F_2$, eq (68) and (69), have exponential terms whose arguments are equal and opposite in sign. When these arguments have magnitude of 2/3 T-lim, they differ in size by 15 decimal places, which is near the 18-decimal-place word size of the machine. The ability to compute the difference in these two terms is essentially the same as the ability to compute the depth function accurately.

## PREVENTING ZEROES IN THE DETERMINANT

Placing limits on $\zeta$ can cause problems in the determinant because $\zeta$ may be set equal to S-lim at several interfaces. The equations that arise for matching boundary conditions may then be identical for these interfaces and may therefore fail to be linearly independent. The triangularized determinant will thus have zeroes on the diagonal at positions equivalent to interfaces that do not have real physical importance for the mode. These will prevent location of the significant "zeroes" or roots. These artificial zeroes must be removed.

The artificial zeroes are detected and removed in the subroutine DET, which evaluates the determinant. If four elements from the matrix have the configuration

a b

c d

and c is to be set to zero by row reduction, d will be replaced by a value, x, as follows:

$$x = d - bc/a.$$

If d is located on the diagonal, complete loss of accuracy is checked for by computing

$$s = |x^2| \Big/ |d^2|.$$

If s is less than $10^{-34}$, x is not used; instead, d is replaced by $10^{-17}$d. Note that this substitution will occur when x is zero, thus preventing zeroes on the diagonal. The power of ten, -17, is chosen to be near the total word size of 18 decimal places.

The above substitution prevents sudden jumps in the value of the determinant when all precision is lost at one step in the evaluation. This is important for the mode search routine which detects roots by looking for minima in a series of values of the determinant while one parameter is incremented slowly. A sudden jump will often produce a relative minimum which will be falsely interpreted as a root. At true roots, one or more elements along the diagonal are small, but not as small as those checked for here.

## REFLECTION COEFFICIENTS AND OTHER AUXILIARY OUTPUTS

Once the depth functions of a mode have been determined, it is relatively easy to compute reflection coefficients at any interface. Therefore, a subroutine called RCOEF has been added to the program which will compute and print out reflection coefficients if requested by the use of control key 3. If key 3 is set to 1, the reflection coefficients at all interfaces are computed. If set to a number, n, greater than 1, the coefficient is computed at the *nth* interface only, where the surface is the first interface.

The printout includes the phase as well as the amplitude of the reflection coefficient and the grazing angle. The grazing angle, $\theta$, of the equivalent rays is computed from the mode phase velocity and the sound speed, c, at the bottom of the layer, by Snell's law:

$$\theta = \cos^{-1}(c/v).$$

The grazing angle is computed only if the phase velocity is greater than the sound speed at the interface, since otherwise the equivalent ray does not reach the interface.

The reflection coefficient is derived, following Bucker (ref 9), by assuming that an isospeed layer exists for a small depth just above the interface. In this layer the depth function can be written as

$$f(z) = Ae^{ilz} + Be^{-ilz}, \tag{43}$$

where l, the vertical component of the mode wave number, is given for mode n by

$$l_n^2 = k_i^2 - \lambda_n^2 \tag{44}$$

and

$$k_i = \omega/c_{bi},$$

where $c_{bi}$ is the sound speed at the bottom of layer i. The derivation now consists of identifying A and B as the pressures of the upgoing and downgoing waves at the bottom of the layer; thus the reflection coefficient

$$R = A/B.$$

A and B are evaluated by making f and its derivative at the interface between this small isospeed layer and the regular profile continuous with the normal mode depth functions. The thickness of the isospeed layer is then allowed to approach zero, giving the desired value of R. If F and F′ are the normal mode function and its depth derivative at the interface depth defined by eq (9) and (19), the reflection coefficient resulting from the above derivation is as follows:

$$R = (ilF + F')/(ilF - F'). \tag{45}$$

This coefficient is a complex number. Loss per reflection is given by 20 times the log of the absolute value. The phase gives the phase shift that an equivalent ray would

9. Sound Propagation in a Channel with Lossy Boundaries, by HP Bucker; J Acoust Soc Am, vol 48, p 1187-1194, November 1970.

experience upon reflection. Figure 3 is an example of the use of this computation. It shows phase and amplitude of the reflection coefficient in shallow water over a sandy-silt sediment lying over rock. The frequency is 1500 Hz. Reflections are given only at discrete points determined by the individual modes.

The model in figure 3 is for a liquid bottom. That is, no rigidity is supplied in this program and the sound speed, density, and attenuation determine the reflection coefficients.

The reflection coefficients computed by eq (45) can be closely approximated by dividing the mode attenuation by the loop length of the corresponding ray. The loop length must be determined from ray theory for the ray of the same phase velocity or vertexing velocity. However, an interesting analog of the ray loop length is the intermode interference length. This is discussed by Guthrie (ref 10). Specifically, if the difference between eigenvalues, $\mathrm{Re}\,\lambda_i$, for two adjacent modes is $\Delta\lambda$, the interference length $l = 2\pi/\Delta\lambda$. This distance will usually equal the ray loop length for some ray with phase velocity between that of the two modes.

As each mode after the first is computed, the length, l, is computed and printed out. Also routinely printed out for each mode is the mode damping or mode attenuation coefficient, in units of dB per km. This attenuation, $\alpha_i$, is computed from the relationship

$$\alpha_i = -1000\ \mathrm{Im}\,\lambda_i \log_{10} e$$
$$= -8686\ \mathrm{Im}\,\lambda_i.$$

This quantity multiplied by range gives the damping of mode i, in dB.

10. The Connection Between Normal Modes and Rays in Underwater Acoustics, by KM Guthrie; J of Sound and Vibration, vol 32, no 2, p 289-293, 1974.

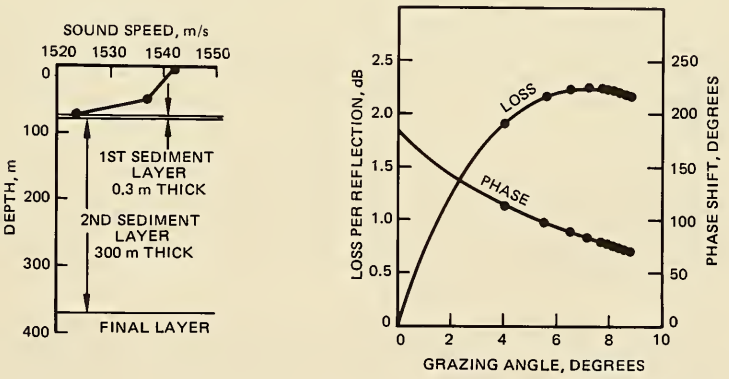


Figure 3. A shallow-water profile with resulting phase and amplitude of the reflection coefficient at 1.5 kHz. Parameters at the top of the sediment layers are as follows: 1st layer – c = 1606.45 m/s, $\gamma = 1.5 s^{-1}$, $\alpha$ = 0.18 dB/m, $\rho = 1.68$; 2nd layer – c = 1684.0 m/s, $\gamma = 1.5 s^{-1}$, $\alpha = 1.10$ dB/m, $\rho = 1.91$; final layer – $\gamma = -0.1$.

## COMPUTATION OF THE MODIFIED HANKEL FUNCTIONS

Most of the computer time required to determine eigenvalues and compute depth functions is spent in evaluating the modified Hankel functions of order 1/3. For this reason, minimizing computer time in evaluating these functions is desirable. Gaining as many places of accuracy as possible is even more important. The average normal mode computation will have many modes that can be determined to far greater accuracy than is required to obtain 0.1 dB accuracy in the propagation loss. However, there are usually some and often many modes in which many places of accuracy are lost in evaluating the determinant. Therefore, maximum accuracy in the modified Hankel functions is required to extend the range of cases for which computations can be carried out successfully.

Optimization of the program is a function of the computer word length. The program given in this report is for the UNIVAC 1110 with 60 bits word length in double precision or 18.1 decimal places. This section gives the equations and computational techniques that are required to optimize this program for different computer word lengths. Complete details of the functions are given in reference 11.

The Airy functions Ai(Z) and Bi(Z) can be used instead of the modified Hankel functions $h_1$ and $h_2$. However, since $h_2$ is ideally suited to matching the boundary conditions at great depth as formulated in this normal mode program, $h_1$ and $h_2$ are used here. The relationship between them is as follows:

$$h_1(z) = k\,[Ai(-z) - i\,Bi(-z)] \tag{46}$$

$$h_2(z) = k^*\,[Ai(-z) + i\,Bi(-z)]\ , \tag{47}$$

where

$k = (3/2)^{2/3}\,(1 - i\sqrt{3}/3)$, and $k^*$ is the complex conjugate of k.

In this section z will be the argument of the functions $h_1$ and $h_2$. For small values of $|z|$, $h_1$ and $h_2$ are computed by power series expansions. For large values, an asymptotic expansion is used. In the past the asymptotic series was expanded directly. However, a continued fraction expansion has been found to give both shorter running time and better accuracy.

Figure 4 shows a line in the complex plane which divides the plane into two parts. For values of z within the line, the power series method is used. When z is outside the line, the continued fraction method is used. This line is a function of computer word length, and the method of determining it will be given after the two methods have been treated. The accuracy of the methods is also treated.

The program has a parameter called IH in the FORTRAN call statement which controls which functions are computed. If IH is set to zero, both functions and their derivatives are computed. If IH is set to 1, only the functions are computed. If it is set to 2, only $h_2$ and its derivative are computed.

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11. Tables of the Modified Hankel Functions of Order One-Third and their Derivatives, Harvard University Computation Laboratory; Harvard University Press, Cambridge MA, 1945.

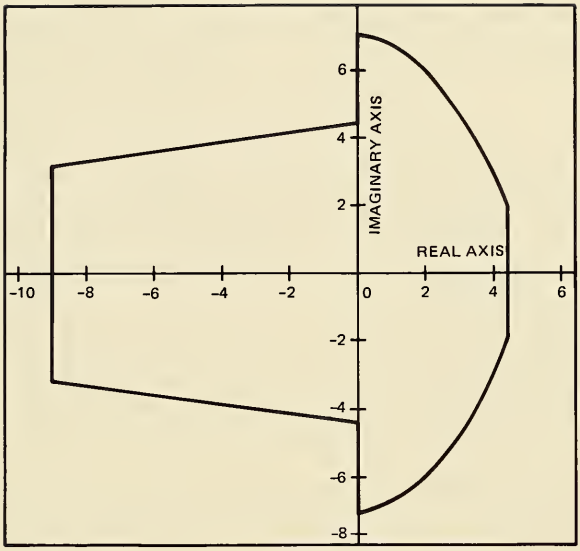


Figure 4. Line in complex plane dividing the arguments for which the modified Hankel functions are computed by (1) power series (inside) and (2) asymptotic expansion evaluated by continued fractions (outside).

## POWER SERIES EXPANSION

In this expansion $h_1$ and $h_2$ are given by two auxiliary functions f and g as

$$h_1(z) = g + i(3)^{-1/2}(g - 2f) \tag{48}$$

$$h_2(z) = g - i(3)^{-1/2}(g - 2f) \ . \tag{49}$$

The auxiliary functions are given by the expressions

$$f = A \sum_{m=0}^{M} a_m X^m \tag{50}$$

$$g = Bz \sum_{m=0}^{M} b_m X^m \ , \tag{51}$$

where $X = z^3$, $A = 2^{1/3}/[\Gamma(2/3)]$ and $B = 2^{1/3}/[3^{2/3}\,\Gamma(4/3)]$. The derivatives $h_1'(z)$ and $h_2'(z)$ can be derived by straightforward differentiation of eq (50) and (51) to give

$$f' = -A z^2 \sum_{m=0}^{M} c_m X^m \tag{52}$$

$$g' = B \sum_{m=0}^{M} d_m X^m \ . \tag{53}$$

The coefficients of eq (50) – (53) are given by recursion relations where $a_0 = 1$, $a_1 = 1/3!$, $a_2 = +1 \cdot 4/6!$, $a_3 = -1 \cdot 4 \cdot 7/9!$

$$a_m = -a_{m-1}/(3m)\,(3m-1) \tag{54}$$

$b_0 = 1$, $b_1 = -2/4!$, $b_2 = +2 \cdot 5/7!$, $b_3 = -2 \cdot 5 \cdot 8/10!$

$$b_m = -b_{m-1}/(3m)\,(3m+1) \tag{55}$$

$c_0 = 3/3!$, $c_1 = -6 \cdot 1 \cdot 4/6!$

$$c_m = -c_{m-1}/3m\,(3m+2) \tag{56}$$

$d_0 = 1$, $d_1 = -4 \cdot 2/4!$

$$d_m = -d_{m-1}/3m\,(3m-2) \tag{57}$$

It is important for efficient computation that the number of terms M be no larger than necessary. In the current program the same value of M is used in all four sums. This is done because the optimum number never differs by more than one in the four cases and the determination by table look-up of four M's often would take longer than computing any unnecessary terms. M for each series is determined so that adding additional terms will not change the answer. Then the most stringent of the four conditions is tabulated and used.

A precise determination of the number of terms to use requires a knowledge of the size of the largest single term in the sum. When a term is smaller than this by a factor which is the power of 10 equal to the number of decimal places in the computer word size, it cannot affect the sum. We ignore the fact that a sum of small terms might be significant. This, then, defines the truncation point. Let m be the number of the largest term in the sum, k the number of terms to be used, and h the number of decimal digits in the machine word. Then for a given k, the largest absolute value of the argument z that can be used to compute g′ is given as

$$|z^3|^m d_m = |z^3|^k d_k \cdot 10^h \ . \tag{58}$$

The power of ten can be replaced by 2 raised to a power of the number of binary bits in the computer word if preferred. The coefficient d of eq (57) is used. Each of the other three should also be tried, to find the smallest number of the group for a given k. Equation (58) can be solved for |z|, giving

$$\log |z| = (\log d_k - \log d_m + h)/(3m - 3k) \ . \tag{59}$$

A simple computer program given in appendix C will find $|z|$ for each value of k from 1 up to the maximum number of terms desired. The largest term, m, is easily determined because from one k to the next m will remain the same or increase by 1, so it is only necessary at each step to check term m+1 to see if it is larger than term m.

The FORTRAN subroutine HANKEL given in appendix A uses the above power series method to compute $h_1$ and $h_2$ for small arguments. The coefficients a, b, c, and d are given in lists by that name. The truncation points are given in the list called ZMLA2, which lists values of $|z|^2$ determined by eq (59) or the three similar equations.

## ASYMPTOTIC SERIES EXPANSION USING CONTINUED FRACTIONS

When the argument z falls outside the curve in figure 4, $h_1$ and $h_2$ can be computed more efficiently or more accurately by asymptotic series than by power series methods. Reference 11 gives information on branch cuts and regions of validity of the two forms of the asymptotic solution (Stokes' phenomenon). Here we will give computing formulas that comply with these requirements, without discussing them further.

Since a given expansion is valid in one or more quadrants, we choose complete quadrants as regions. For z in quadrants 1, 3, or 4 use

$$h_2(z) \sim \exp(5\pi i/12)\, F_2(z) \tag{60}$$

$$h_2'(z) \sim \exp(-\pi i/12)\, G_2(z) \tag{61}$$

For z in quadrant 2 use

$$h_2(z) \sim \exp(5\pi i/12)\, F_2(z) + \exp(11\pi i/12)\, F_1(z) \tag{62}$$

$$h_2'(z) \sim \exp(-\pi i/12)\, G_2(z) + \exp(-7\pi i/12)\, G_1(z) \tag{63}$$

For z in quadrants 1, 2, or 4 use

$$h_1(z) \sim \exp(-5\pi i/12)\, F_1(z) \tag{64}$$

$$h_1'(z) \sim \exp(\pi i/12)\, G_1(z) \tag{65}$$

For z in quadrant 3 use

$$h_1(z) \sim \exp(-5\pi i/12)\, F_1(z) + \exp(-11\pi i/12)\, F_2(z) \tag{66}$$

$$h_1'(z) \sim \exp(\pi i/12)\, G_1(z) + \exp(7\pi i/12)\, G_2(z) \tag{67}$$

The four auxiliary functions follow:

$$F_1(z) = K z^{-1/4} \exp(2i z^{3/2}/3) \sum_{m=0}^{M} C_M X^m \tag{68}$$

$$F_2(z) = k\, z^{-1/4} \exp(-2i\, z^{3/2}/3) \sum_{m=0}^{M} C_m Y^m \tag{69}$$

$$G_1(z) = k\, z^{1/4} \exp(2i\, z^{3/2}/3) \sum_{m=0}^{M} D_m X^m \tag{70}$$

$$G_2(z) = k\, z^{1/4} \exp(-2i\, z^{3/2}/3) \sum_{m=0}^{M} D_m Y^m \tag{71}$$

where X and Y equal $\mp i\, z^{-3/2}$ respectively, and

$$K = 2^{1/3}\, 3^{1/6}\, \pi^{-1/2} = 0.853\,667\,218\,838\,951$$

The coefficients $C_m$ and $D_m$ are again computed by recursion relations where $C_0 = D_0 = 1$:

$$C_m = C_{m-1} [9(2m-1)^2 - 4]/48m \tag{72}$$

and

$$D_m = D_{m-1} [9(2m-1)^2 - 16]/48m\ . \tag{73}$$

Square roots of z are to be taken so that the real part of the root is always positive and the imaginary part has the same sign as the imaginary part of z. This applies also to fourth roots. The three-halves power is obtained as the product of z and its square root.

The summations in eq (68) – (71) can be done as indicated or evaluated by continued fractions. When done as indicated they are asymptotic series, and care must be taken to truncate them at the term of smallest magnitude, if this term is reached, because adding more terms will reduce the accuracy. Since the largest term in these series will always be 1, the series can be truncated if the terms become less than $10^{-h}$ in magnitude, where h is the number of decimal digits in the computer word.

**Continued Fraction Expansion**

The method of continued fractions is more effective in evaluating these asymptotic series, and it is used in subroutine Hankel in the FORTRAN program in this report. The coefficients are stored in lists entitled C4, C5, D4, and D5. In the remainder of this section the continued fraction technique is presented, along with the method of determining coefficients.

The continued fraction has the form

$$F(x) = b_0 + \cfrac{a_1}{x + b_1 + \cfrac{a_2}{x + b_2 + \ldots}}\ . \tag{74}$$

It is to be used to evaluate a polynomial

$$P(x) = \sum_{m=0}^{M} C_m X^m \quad . \tag{75}$$

This polynomial can represent any of eq (68) – (71). One of three standard forms for continued fractions, this form is used because it has two coefficients at each stage and therefore is equivalent to an asymptotic series of twice as many terms. This reduces by half the number of divisions required. Since complex divisions are lengthy, requiring six real multiplications and two divisions, this is the only standard form of the continued fraction that can compete in computer time with the asymptotic series.

The coefficients $a_i$ and $b_i$ of eq (74) must be determined from the coefficients $C_m$. The usual technique is to express P as a rational function, then use the continued fraction to evaluate the rational function. The determination of the coefficients can be done in these two steps or by a second method which goes directly from power series to continued fraction coefficients. The second method is preferable because the loss of accuracy is more in the first. But since the first method is more easily understood, each method will be given; a computer program is included in appendix C which will determine coefficients by the second method.

Let M in eq (75) be an even number so that 2N = M. (An additional unnecessary term of the series can always be used.) The rational function will have the form

$$R(x) \approx k \sum_{i=0}^{N} \hat{e}_i x^i \Big/ \sum_{i=0}^{N} \hat{f}_i x^i \quad , \tag{76}$$

where $\hat{e}_0 = \hat{f}_0 = 1$ and $k = C_0$. The coefficients $\hat{e}_i$ and $\hat{f}_i$ are evaluated from a set of linear equations which can be described by displaying a particular case. For N = 3 they are as follows:

$$\begin{bmatrix} -1 & 0 & 0 & C_0 & 0 & 0 \\ 0 & -1 & 0 & C_1 & C_0 & 0 \\ 0 & 0 & -1 & C_2 & C_1 & C_0 \\ 0 & 0 & 0 & C_3 & C_2 & C_1 \\ 0 & 0 & 0 & C_4 & C_3 & C_2 \\ 0 & 0 & 0 & C_5 & C_4 & C_3 \end{bmatrix} \begin{bmatrix} k\hat{e}_1 \\ k\hat{e}_2 \\ k\hat{e}_3 \\ \hat{f}_1 \\ \hat{f}_2 \\ \hat{f}_3 \end{bmatrix} = - \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{bmatrix} \tag{77}$$

With $e_i$ and $f_i$ thus determined, R(x) is equivalent to P(x) through the first M + 1 terms. R(x) can now be evaluated exactly (except for round-off error) using a continued fraction of the form F(x) of eq (74).

Rather than R(x), however, a similar expression in y = 1/x is the form that is well suited to evaluating asymptotic series. This expression is obtained by dividing each term of R(x) by $X^N$. The order of the coefficients can now be reversed and a simple algebraic operation can yield a value of 1 for each of the two initial coefficients and a new value for k. We will call this new rational function with renamed coefficients $\hat{R}(y)$. It will have the form of eq (76) but different coefficients, say e and f instead of $\hat{e}$ and $\hat{f}$.

The coefficients $a_i$ and $b_i$ are determined from $e_i$ and $f_i$ by a recursive formula which involves constructing an n × n triangular matrix Q with elements $q_{i,j}$ as follows:

$$b_0 = e_0$$

$$q_{1,i} = (e_i - e_0 f_i)/a_1 \qquad i = 1, 2, \ldots, N ,$$

where $q_{1,1} = 1$, giving $a_1$, and

$$b_1 = f_1 - q_{1,2} .$$

The second row:

$$q_{2,i} = (f_i - q_{1,i+1} - b_1 q_{1,i})/a_2 \qquad i = 2, 3, \ldots, N ,$$

where $q_{2,2} = 1$, giving $a_2$, and

$$b_2 = q_{1,2} - q_{2,3} .$$

Elements outside the matrix are assigned a value of zero. The remaining rows for m = 3 to N are as follows:

$$q_{m,i} = (q_{m-2,i} - q_{m-1,i+1} - b_{m-1} q_{m-1,i})/a_m \qquad i = m, m+1, \ldots, N ,$$

where $q_{m,m} = 1$, giving $a_m$, and

$$b_m = q_{m-1,m} - q_{m,m+1} .$$

The second method determines the continued fraction coefficients $a_i$ and $b_i$ directly from the asymptotic series coefficients $c_i$. This method is preferable to the first because the loss of accuracy in inverting the matrix in eq (77) can be more than the loss in this second method.

It has been pointed out* that the second method is probably a variant of the Viskovakoff algorithm described by Khovanskii (ref 12) and as such is unstable – subject to accumulation of errors. However, it is sufficiently stable to obtain the required coefficients.

*Private communication with AN Stokes, CSIRO, Wembley, Western Australia.

12. The Application of Continued Fractions and their Generalizations to Problems of Approximate Analysis, by AN Khovanskii; a monograph in Russian, 1956.

The coefficients are derived as follows. The well-known recursive relations that give the N*th* stage of a continued fraction as a rational function are used (ref 13).

$$F_N(y) = A_N(y)/B_N(y) \ , \tag{78}$$

where

$$A_N = \sum_{i=0}^{N} e_i y^i$$

$$= (y + b_N) A_{N-1} + a_N A_{N-2} \tag{79}$$

$$B_N = \sum_{i=0}^{N} f_i y^i$$

$$= (y + b_N) B_{N-1} + a_N B_{N-2} \ , \tag{80}$$

in which $A_{-1} = 1$, $A_0 = b_0$, $B_{-1} = 0$, and $B_0 = 1$. Again $y = 1/x$. The long division indicated in eq (78) is then carried out, giving a quotient in terms of $a_i$, $b_i$, and y that can be equated, term by term, to the first 2N−1 terms of the asymptotic series.

The long division is carried out with $A_N$ and $B_N$ written in descending powers of y. The quotient is then in descending powers of y or ascending powers of x. Fortunately, the first 2N+1 terms determined for any N are identical to the same initial terms for any larger value of N. This will be proven later. The first few equations obtained from the division are as follows:

$$b_0 = C_0$$

$$a_1 = C_1$$

$$-a_1 b_1 = C_2$$

$$a_1 \left( b_1^2 - a_2 \right) = C_3$$

$$a_1 \left( 2 a_2 b_1 - b_1^3 + a_2 b_2 \right) = C_4 \tag{81}$$

From these equations $a_i$ and $b_i$ can be determined, since the coefficients $C_i$ are known. However, a simpler method is available.

The long division indicated in eq (78) can be carried to 2N+1 valid places; but beyond N+1 places, terms from the original dividend are no longer entering the remainder. Therefore terms in the later part of the quotient have a simplified form. Since term n+1 of

13. Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, ed by M Abramowitz and IA Stegun; National Bureau of Standards Applied Mathematics Series, vol 55, p 19, 1964.

the quotient is equal to the *nth* asymptotic coefficient, designate it $C_m$. Note that the C's are numbered from 0 to N. Let the coefficient of $y^m$ in $B_n$ be $B_{n,m}$. Then

$$C_{N+j} = -\sum_{i=1}^{N} B_{N,N-i}\, C_{N+j-i} , \qquad 1 \leqslant j \leqslant N . \tag{82}$$

Here the C's are numerical constants. The unknowns, the a's and b's, are in the terms of B. Suppose that these unknowns have been determined up to n = N-1. Then eq (82) will contain two unknowns, $a_N$ and $b_N$. By using eq (82) for j = N-1 and N, the unknowns can be evaluated. The index N can then be increased by 1 and the process repeated. The process can start with N = 2 if $a_1$, $b_0$, and $b_1$ are provided, but these are easily determined from eq (81). The terms of $B_N$ are determined from eq (80), which gives for each term

$$B_{n,m} = B_{n-1,m-1} + b_n B_{n-1,m} + a_n B_{n-2,m} . \tag{83}$$

Any $B_{n,m}$ is zero if m is greater than n.

When j = N-1 is used in eq (82) in the process described above, the coefficient of the (2N-1) power of x is being evaluated. This term is expected to contain $a_N$ and $b_N$, but – as will be proven later – because the coefficient of $b_N$ is zero, $a_N$ is the only unknown in a linear equation and can be easily evaluated. The next term determined with j = N contains $a_N$ and $b_N$, but now only $b_N$ is unknown and is easily evaluated.

As an example, the $C_n$'s through n = 10 are listed in table 1. These are the asymptotic series coefficients given by eq (72). The corresponding $a_n$'s and $b_n$'s as determined above are also listed. A more complete list of the a's and b's can be obtained from the FORTRAN program in appendix C.

Table 1. Asymptotic series coefficients, $C_n$, and the corresponding continued fraction coefficients, $a_n$ and $b_n$.

| n | $c_n$ | $a_n$ | $b_n$ |
|---|---|---|---|
| 0 | 1. | 0 | 1. |
| 1 | 0.10416 | 0.10416 | -0.80208 |
| 2 | 0.08355 | -0.58764 | -2.28555 |
| 3 | 0.12823 | -2.29072 | -3.77864 |
| 4 | 0.29185 | -5.11525 | -5.27462 |
| 5 | 0.88163 | -9.06285 | -6.77193 |
| 6 | 3.32141 | | |
| 7 | 14.99576 | | |
| 8 | 78.92301 | | |
| 9 | 474.45154 | | |
| 10 | 3207.49009 | | |

A FORTRAN program to compute the continued fraction coefficients for the series given by eq (72) is given in appendix C. This program can be easily modified to determine the other set.

**Two Proofs**

In this section proofs will be given of two facts used in the previous section. Following this, the number of terms required, the accuracy, and similar topics will be discussed. To prove that the first 2N+1 terms of the quotient $A_N/B_N$ are equal to the same terms when N is a larger integer, use long division on eq (79) and (80) to obtain

$$A_N/B_N = A_{N-1}/B_{N-1} + a_n (A_{N-2} B_{N-1} - A_{N-1} B_{N-2})/(B_N B_{N-1}) \quad . \tag{84}$$

If the first quotient on the right is to have terms equal to the quotient on the left up through term 2N-1, the remainder must have no terms with y to a higher power than -(2N-1). The final divisor, $B_N B_{N-1}$, contains y to the (2N-1) and lower powers. Therefore, the proof is complete if the numerator of the remainder is a constant. To show this, use eq (79) and (80) to evaluate $B_{N-1}$ and $A_{N-1}$; it can be shown that

$$A_{N-2} B_{N-1} - A_{N-1} B_{N-2} = -a_{N-1} (A_{N-3} B_{N-2} - A_{N-2} B_{N-3})$$

$$= (-1)^N a_{N-1} a_{N-2} \cdots a_1 \quad .$$

The right-hand product is obtained by repeatedly applying the middle result. The product of a's is a constant, completing the proof.

The second proof required is that in the quotient of $A_N/B_N$, $C_{2N}$ (the coefficient of $y^{-2N}$) will contain no $a_i$ or $b_i$ to higher than term N and $C_{2N+1}$ (the coefficient of $y^{-2N-1}$) will involve no $a_i$ to higher than term N+1 and no $b_i$ to higher than term N.

The first part is intuitively obvious. Since from the preceding proof $C_{2N}$ will be the same when derived from the ratio $A_x/B_x$ for any x as long as it is N or greater, we need consider only the case where x is N. But since from eq (79) and (80) $A_N$ and $B_N$ contain no a's or b's of greater than term N, $C_{2N}$ cannot contain a's or b's of higher terms.

By the same argument $C_{2N+1}$ can contain no a's or b's to higher terms than N+1. There remains to be proven only that $b_{N+1}$ cannot exist in $C_{2N+1}$ or that its coefficient, which we will call E, is zero. Applying eq (82) for N+1 and j = N gives

$$C_{N+1+N} = - \sum_{i=1}^{N+1} B_{N+1,N+1-i} C_{N+1+N-i} \quad .$$

E, the coefficient of $b_{n+1}$ in this expression from eq (83), takes the form

$$E = - \sum_{i=1}^{N+1} B_{N,N+1-i} C_{2N+1-i} \quad .$$

But by choosing j = N in eq (82) we see that terms 1 to N for $C_{2N}$ equal terms 2 to N+1 in E, so

$$E = -B_{N,N} C_{2N} + C_{2N} .$$

However, since $B_{N,N}$, the coefficient of $y^n$ in $B_n$, is always 1 by eq (80), E = 0. Therefore $b_{N+1}$ does not exist in $C_{2N+1}$.

### Number of Terms

The number of terms or stages to use in the continued fraction was arrived at by a trial and error process. For a given number of terms, a real positive argument was decreased until the accuracy began to drop. The magnitude just before this drop was considered to be the optimum point to increase the number of stages by one. Because the argument to the continued fractions is $z^{3/2}$, we took the larger of the magnitudes of the real and imaginary parts of $z^{3/2}$ as the test number. This number is then compared to the 3/2 power of the points determined along the real axis by trial and error.

The above method appears to work well although it involves no thorough understanding of the way complex numbers affect the successive convergents of a continued fraction. Table 2 shows the points down to which a given number of stages gives full accuracy for positive real arguments and lists the 3/2 power of these numbers as used in the FORTRAN program list called ZMLA5.

### Division Lines

The power series method is now to be used for small arguments and the continued fraction method for large arguments. The exact dividing line between them is needed. The division line of figure 4 was arrived at by computing the functions along rays from the origin, using both power series and continued fractions. The number of decimal places to which the functions determined by the two methods agree tends to reach a maximum at some distance from the origin along each ray. At distances short of this maximum we can assume that the continued fraction method is less accurate than the power series. At distances beyond the maximum, the power series is assumed to be less accurate. The maximum therefore indicates the ideal place to change from one method to the other if the decision is to be based solely on accuracy. This method was used to determine figure 4.

A complication arises, however. Along certain rays from the origin, $h_1$ and its derivative reach a maximum number of places at very different distances from $h_2$ and its derivative. The principal problem is at ±60° but persists from about 30° to 90°. At 60°, $h_1$ is small in magnitude and $h_2$ is large. The power series method cannot compute the small values accurately due to loss in accuracy in subtraction in eq (48). The accuracy of the continued fraction for $h_2$ is poor at 60° because eq (69) becomes a nonalternating series and continued fraction approximations are not known to improve the accuracy of nonalternating asymptotic series as they do for alternating series.

A reasonable solution to this problem is to compute $h_1$ by continued fractions and $h_2$ by power series for arguments at these angles and magnitudes from 4 to 10. However, as will be shown later, the above solution has not been employed at this time since this area is not of great importance for normal mode computations. Instead, the argument was chosen

Table 2. Cut-off points for determining the number of stages in the continued fractions.

| Number of Stages | Real Argument x | Program Test Value $x^{3/2}$ |
|---|---|---|
| 1 | $10^6$ | $10^9$ |
| 2 | 80 | 715.0 |
| 3 | 35 | 207.0 |
| 4 | 22 | 103.0 |
| 5 | 13 | 47.0 |
| 6 | 11 | 36.4 |
| 7 | 9 | 27.0 |
| 8 | 8 | 22.6 |
| 9 | 7 | 18.5 |
| 10 | 6.5 | 16.6 |
| 11 | 6 | 14.7 |
| 12 | 5.8 | 14.0 |
| 13 | 5.5 | 12.9 |
| 14 | 5.3 | 12.2 |
| 15 | 5.1 | 11.5 |
| 16 | 4.9 | 10.8 |
| 17 | 4.5 | 9.5 |
| 18 | 4.4 | 9.2 |

that gave equivalent accuracy for the two methods. Along 60° this minimum accuracy is 9 decimal places.

The following relationship exists between $h_1$ and $h_2$ for positive and negative values of the imaginary part of the arguments:

$$h_1(z^*) = [h_2(z)]^* ,$$

where the * means complex conjugate. Thus, the above discussion at 60° can be translated to -60°. Also, the functions actually need to be computed only in quadrants I and II. They could then be evaluated in quadrants III and IV by the above relationship. The above relationship explains the symmetry of figure 4 about the real axis.

## COMPARATIVE ACCURACY

The accuracy of the three methods – power series, asymptotic series and continued fractions – has been determined on a CDC computer with 48 bits or 14.4 decimal places of accuracy in the floating point word. Since this differs from the double precision word length of 60 bits or 18.1 decimal places that applies to the preceding part of this report, these results are for comparative and illustrative purposes only.

Accuracy is determined by computing the functions and either comparing the answers for the several different computing methods or computing the wronskian. The wronskian is a constant given by the relationship

$$h_1 h_2' - h_2 h_1' = -1.45749544104i = -i\,96^{1/3}/\pi \ . \tag{85}$$

The wronskian will determine the accuracy of the functions if it can be computed without loss of accuracy. If the two products in it are large, though, accuracy will be lost in the subtraction. This generally happens for arguments near the negative real axis. Here accuracy must be determined by comparing answers from different methods. The accuracy of the functions and their derivatives will generally be about equal.

Figure 5 illustrates the accuracy that is obtained in different parts of the complex plane of the argument, z, by using the power series method. On the inner contour, the functions $h_1$ and $h_2$ and their derivatives have 12 places of accuracy. On the outer contour, the accuracy is 11 places. As expected, the accuracy is best for arguments of small magnitude. The accuracy remains best in directions from the origin in which the functions are large in magnitude. This is because less accuracy is lost in subtraction. Accuracy must be lost when individual terms of the series are large but the sum is small.

Figure 6 shows accuracy contours for the asymptotic expansion with both the direct and continued fraction evaluation of the series. Here, the best accuracy is obtained for large arguments, and accuracy decreases toward the origin. As can be seen, each of the two methods is better in some directions from the origin. The choice of methods then depends upon which directions are of most value to the normal mode program. The dots on the figure show the locations at which the functions were evaluated in a typical surface duct run. Although arguments can lie anywhere in the plane, most of them follow this pattern. They lie just above the negative real axis and in a narrow angle above the positive real axis. The continued fraction method is distinctly better on this positive side. Since computing time also favors the continued fraction method, it is clearly the method to use.

If the 12-place accuracy contour from figure 6 lies inside that for figure 5 at some angle from the origin, 12 places can be obtained at any range along this angle by using either power series or asymptotic expansion in the interval of overlap. If the asymptotic expansion contour lies outside the other, there is an interval in which 12 places cannot be obtained. Only some lesser number of places can be obtained in this interval. These contours apply when both functions and their derivatives are all computed by a single method. As mentioned earlier, increased accuracy could be obtained in some areas by computing the two functions by different methods.

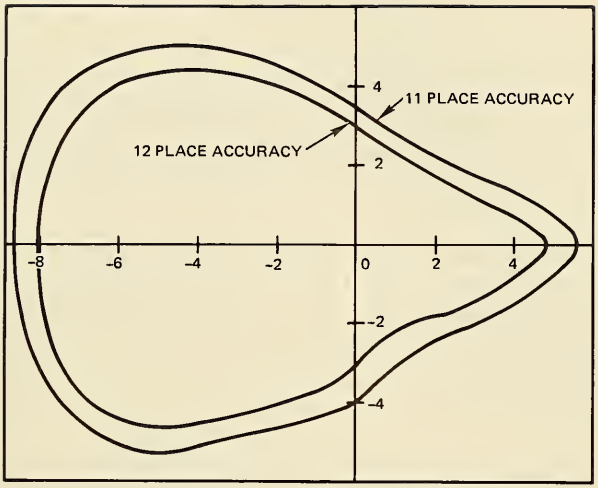


Figure 5. Locus of arguments for which the power series evaluation of the modified Hankel functions gives 12 and 11 decimal places of accuracy for a computer word length of 14.4 decimal places.

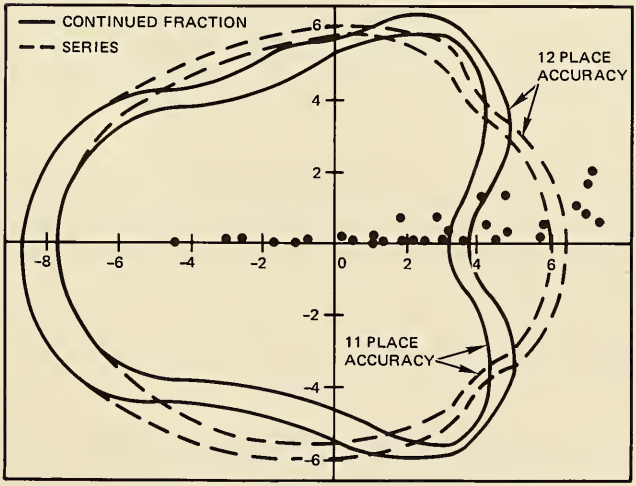


Figure 6. Locus of arguments for which the direct and continued fraction evaluation of the asymptotic series gives 11 and 12 decimal place accuracy. The arguments at which the modified Hankel functions were evaluated in a typical normal-mode run are shown.

## MODE FOLLOWER PROGRAM

Appendix D lists the Mode Follower Program in FORTRAN. It is not a part of the general normal mode program, but is related in that it uses some parts of the general program. The purpose of the mode follower is to trace a given eigenvalue as some parameter is varied. This parameter is usually frequency, but any profile parameter can also be varied. The eigenvalues at a given set of parameters are discrete points. By permitting the parameter to vary, the eigenvalues become a set of lines, and this often clarifies their behavior at the fixed points. Figures 7–9 illustrate this.

Figure 7 is a sound speed profile consisting of two ducts. Figures 8 and 9 show the real and imaginary parts of some eigenvalues of the profile over a range of frequencies. The imaginary parts are expressed as mode attenuations. The figures show a region where both ducts are exerting an influence on the eigenvalues. The broken lines show the location of eigenvalues for a profile that consists of only the upper duct of figure 7. Considerable time could be spent studying the interaction between the two ducts, but since the purpose here is to illustrate eigenvalues as functions of a parameter, only a brief description will be given.

Modes are numbered by the real parts of their eigenvalues. This numbering is consistent with the number of beats or changes of $\pi$ in the phase of the depth functions. Thus the eigenvalue of a mode numbered 1 in a profile consisting of only the upper duct lies exactly over the eigenvalues of a mode in the double duct in figures 8 and 9, but this mode in the double duct changes number each time it crosses the real part of another mode. The depth function actually gains an additional beat each time this happens. The background of modes that are being crossed consists of the higher order, untrapped modes associated with the lower duct.

Mode 2, of the upper duct only, does not have a single mode in the double duct that overlies it exactly. Instead, a mode attempts to follow it at frequencies above 1350 Hz. Below this frequency, successive modes follow its path for short intervals. This interplay between modes occurs when mode 2 of the upper duct is in some sense equally as untrapped as the modes associated with the lower duct.

The imaginary parts of the modes follow similar patterns; but because the mode numbering is not determined by the imaginary parts, the mode numbers sometime jump from one line to another. An important feature of these two plots is that if the real parts of the eigenvalues cross, the imaginary parts do not; and vice versa. Thus two eigenvalues do not tend to become equal at a point which would make them degenerate.

The mode follower program will tend to follow the continuous curves. Thus if started in the right direction on mode 59 at 1450 Hz, it will follow along the continuous mode which becomes successively mode 58, 57, 56, and 55.

Figures such as 8 and 9 can be drawn by computing the eigenvalues at a sufficient number of frequencies to determine the lines. The mode follower does this for a given eigenvalue while adjusting the step size so the mode will not be lost, or so the program will correctly follow the mode. The step size is permitted to become large where the eigenvalue can be approximated by a parabolic curve, but it shortens when extrapolation to the next point becomes less accurate.

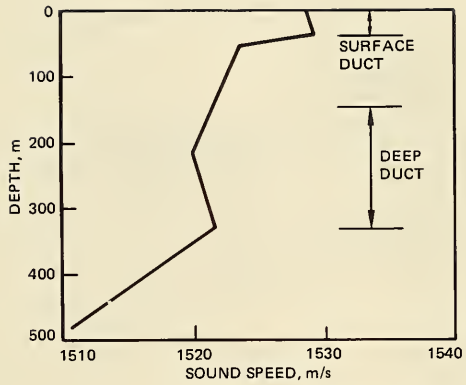


Figure 7. A five-layer approximation to the sound speed of a surface duct overlying a refractive duct.

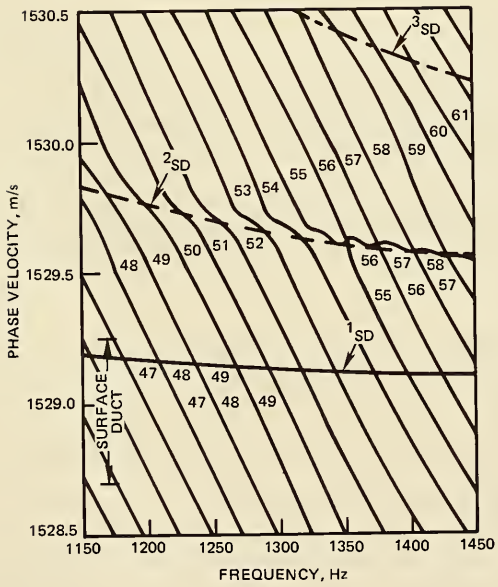


Figure 8. The real part of the eigenvalues as continuous functions of frequency for the profile of figure 7. Some mode numbers are given. The first three modes for the surface duct only (SD) are shown as broken lines. That for mode 1 coincides with an existing line.

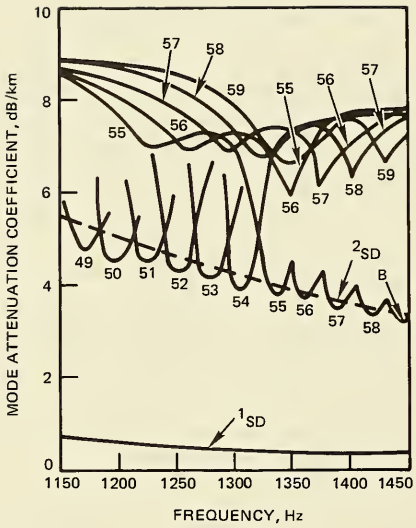


Figure 9. The imaginary parts of the modes whose real parts are shown in figure 8 expressed as mode attenuation. To avoid confusion, they are not shown across the full frequency interval.

When frequency is the variable parameter, the group velocity of the mode can be computed easily since a numerical derivative can be computed. Group velocity is given by the relationship

$$C_g = d\omega/d\,(\mathrm{Re}\,k)$$

$$\cong \Delta\omega/\Delta\,(\mathrm{Re}\,k)$$

$$\cong -\Delta f\, v^2/f\,\Delta v\ ,$$

where k is the horizontal wave number of the mode and v is the real phase velocity. The mode follower prints this value out at each step, along with the eigenvalue.

## IMPLEMENTATION OF THE MODE FOLLOWER

The mode follower was originally implemented for a two-layer normal mode which differed from the n-layer program in that the derivative dG/dv of the characteristic equation was evaluated along with G. The iteration for roots of G was thus Newton-Raphson and is given by the relationship

$$v_{i+1} = v_i - G/G'\ . \qquad (86)$$

This is simpler than the secant iteration of eq (15), in which G′ must be evaluated numerically. Because of the simpler iteration, an effective scheme for mode following was available.

Since considerable effort was required to develop a similar scheme for the n-layer case, the two-layer mode follower will be briefly described to serve as an introduction to the n-layer case.

The two-layer mode follower employs one iterative step of eq (86) at each point where G is evaluated. Thus, a root that is inexact but sufficiently exact is obtained. The original estimate is obtained by extrapolating from the three most recent roots. If this estimate is sufficiently close to the true root, the single iterative step will make a small correction, G/G', that will bring the estimate very close to the true root. By using the size of this correction to control the step size, the program is self-regulating. The program works well when a permissible value of G/G' of $10^{-6}$ to $10^{-4}$ is used. Outside this interval the step size is either doubled or halved.

The multiple-layer program differs from this in several details. The extrapolation from the previous three points is done not only for the phase velocity but also for the numerical derivative,

$$D^{-1} = \Delta v/\Delta G.$$

Lagrange three-point interpolation is used, given by the form

$$v(x) = \frac{v(x_1)(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} - \frac{v(x_2)(x-x_1)(x-x_3)}{(x_1-x_2)(x_2-x_3)} + \frac{v(x_3)(x-x_1)(x-x_2)}{(x_1-x_3)(x_2-x_3)}, \tag{87}$$

where x is the new value of the parameter that is being varied (usually frequency) and $x_1$, $x_2$, and $x_3$ are the three previous values, $x_1$ being the most recent. To extrapolate the derivative, v is replaced by $D^{-1}$ in eq (87). Both quantities are complex numbers.

The determinant is now evaluated at this new phase velocity to give a value $G_0$. Next a corrected value of phase velocity, $v_0$, is obtained:

$$v_0 = v - G_0 D^{-1} . \tag{88}$$

In the two-layer case, the size of the correction, $GD^{-1}$, is used to control the step size. Because the numerical derivative is less precise, we evaluate G once more at this new position, obtaining $G_1$. A new numerical derivative is next calculated:

$$D_0^{-1} = (v_0 - v)/(G_1 - G_0) .$$

This derivative is now compared with the extrapolated value to determine whether the step size should be changed. To do this an error

$$E = |1 - D_0/D|^2$$

is computed. Good results have been obtained by keeping E between $10^{-5}$ and $10^{-2}$. If E becomes larger than this, the step size is halved and the extrapolation is tried again. Should halving the step size five times fail to obtain a value of E less than $10^{-2}$, the mode is presumed to be lost and the program halts.

If E is less than $10^{-2}$, the step is successful and the stored values are updated for the next step. Before v is stored, though, the iterative step of eq (88) is applied one more time to obtain a more precise value of v. This requires little extra effort because the quantities $G_1$ and $D_0^{-1}$ are already available.

If the error E is less than $10^{-5}$, the next step size is doubled.

It is possible for the extrapolation to be too successful. That is, if v is very near the true root, $G_0$ and $\Delta v$ will be very small. The numerical derivative may then be inaccurate. Therefore, before the error term E is computed, a quantity

$$F = |v/\Delta v|^2$$

is computed. If F is greater than $10^{28}$, the extrapolated derivative is used rather than the computed derivative and the program proceeds to the next step. If F is greater than $10^{34}$, the step size is doubled before proceeding to the next step.

The other principal part of the program is the initialization which must evaluate v at three values of x to obtain the numbers needed for the first extrapolation, eq (87).

## INPUT AND OUTPUT

The first input card contains the maximum number of steps allowed, the limits applied to E and F, and keys which control both the amount of detail in the printout and whether the profile parameters are to be read in or retained from the previous run. Default values are supplied when these items are left blank. Next the profile parameters are read in. These are an older style and only permit specification of the absorption loss at the top of a layer. The sound speed gradient is assumed to be real at the top of any layer.

A final card indicates which variable – frequency, sound speed, depth, gradient, absorption, or density – will be varied, by specifying a number called nx in the program, from 1 to 6. The next number, ny, specifies which layer the variable will be in. This layer number is not needed if frequency is selected. A third number, nz, indicates, if zero, that the profile will remain continuous as the selected parameter is varied. If nz is not zero, the selected parameter moves alone without a compensating motion in other profile parameters. The card next gives the initial and final value of the parameter to be varied and the initial step size. Finally, the particular mode to be followed is indicated by giving an approximate phase velocity and an initial step size. These must be chosen such that the subsequent iteration will converge on the correct mode.

The principal output of this program is the print statement at line 314. Each line of output contains the value of the parameter being varied, the complex phase velocity, the determinant G, the derivative $D^{-1}$, the error term E, the mode attenuation, the mode group velocity (if frequency is the parameter being varied), and the step number. After the final step, the profile in its final form is printed out.

## CONCLUSIONS

1. An effective program for computing propagation loss in a layered ocean by normal modes has been developed. Complete documentation for the program is contained herein.

2. Sediment layers are modeled as fluids in which densities, sound speeds, and absorption can be specified. This permits a complete wave solution for bottom reflected sound energy.

3. A continued fraction technique for evaluating asymptotic series is shown to give superior results in evaluating the auxiliary functions required in this program, the modified Hankel functions of order 1/3.

4. A mode follower program given here is useful in tracing eigenvalues. Such traces are needed to understand the eigenvalue structure.

## RECOMMENDATIONS

1. Improve the mode locating ability of this normal-mode program to make it self-contained. It currently requires user interaction to locate eigenvalues.

2. Investigate methods to incorporate the effect of rough boundaries into this program.

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## APPENDIX A: NORMAL MODE PROGRAM IN FORTRAN

This program consists of the main program and seven subroutines. The main program handles the input and output and performs much of the computation. This includes profile preparation, mode search, determination of depth function coefficients, normalization, computation of depth functions, and summation of modes. Auxiliary functions are performed by the subroutines SETUP and DET, which set up the determinant, then evaluate it. This is the determinant from which eigenvalues are determined. The subroutine HZERO determines the Hankel functions of order zero, second type, which gives the range dependence of the modes. Only a single term of the asymptotic expansion is needed for this function.

Subroutine HANKEL evaluates the modified Hankel functions of order 1/3, by which the depth dependence of the modes is expressed. The majority of computing time is usually expended in this subroutine. Subroutine CFR is used by subroutine HANKEL to evaluate continued fractions. Subroutine RCOEF evaluates and prints reflection coefficients when they are requested.

```
      C     THIS IS THE MAIN PART OF  NLAYNM
            IMPLICIT DOUBLE PRECISION (A-H,O-Z)
            DOUBLE PRECISION  LAMBDA,LAMBDI
            INTEGER COL
            REAL R ATTEN, T RE, RX
            DIMENSION LOSPCH(5,25)
            COMMON /HAN/ H2R,H2I,H1R,H1I,H2PR,H2PI,H1PR,H1PI,R
            COMMON/INPUT/ Z(12), N, OMEGA, V, VI, CON(12), GSQ(12),
           1 CAY(12), LAMBDA, LAMBDI, G(12)
           2,RHO(12), GI(12), G SQI(12), CAYI(12)
            COMMON /EXPO/ EXSUM, CNTR, RATIO(25)
            COMMON/DETMNT/ A(25,4), Q(25,4)
            COMMON/PARTS/ ZT(12),ZTI(12),ZB(12),ZBI(12)
            COMMON/REFL/ AF(12,200), AG(12,200), BF(12,200), BG(12,200),
           2 EIGEN(350), EIGENI(350), B (25,4), BI(25,4), CB(12), CBI(12),
           3 CAYSQ(12), CAYSQI(12), NN
            DIMENSION D(350), DI(350), F(100), FI(100), HZERO2(350),
           * DA(350), SRES(350), GAMMAI(12), BLPK(12),
           4 HZER2I(350), DPK(12), GCU(12), GCUI(12), CI(12),
           3 PHASE V(350), PHASI V(350), UU(2000), UUI(2000)
            COMMON /LIMIT/ TLIM, EXPONT, SLIM
            DIMENSION LOSS(101)
            DIMENSION C(12), DEPTH(52), DBLOSS(350), COL(120),
           1CONTR(10), EF(2), FMAG(350), FANG(100),
           2GAMMA(12),JSMBL(10),JCOUNT(5),JCOU(5),LEVEL(41),PLEV(5),RLOSS(100)
           3 , RLOS(101),RECVRS(51),TEST(3),       ING(11)
            EQUIVALENCE  (FF,EF(1)),(DEPTH(1),SOURCE),(DEPTH(2),RECVRS(1)),
           1                        (RLOS(2),RLOSS(1))
            COMMON /AHZERO/ HZEROR,HZEROI
            DATA ( CONTR(I), I=1,4) /110.D0,95.D0,80.D0,-1000.D0/,
           1  (J SMBL(I), I=1,3) /1H1,1H*,1HB/,
           *(ING(I), I = 1,10)/1H0,1H1,1H2,1H3,1H4,1H5,1H6,1H7,1H8,1H9/,
           2  (TEST(I), I=1,3) /.2D0,1.D0,5.D0/
            TLIM = 25.
            SLIM = -8.54988
      C  READ IN PARAMETERS
       1     READ 11, K1, K2, K3, K4, K5, K6, K7, K8, K9
      C KEYS& 1-DEPT FN PRINTOUT, 2-LOSS PRINTOUT, 3-REFLECTION COEFF PRINTOUT
      C  4-CHANGE CONTOURS, 5-CONTINUE MODES
       11    FORMAT (10I4)
             PRINT 13, K1, K2, K3, K4, K5, K6, K7, K8, K9
       13    FORMAT (6H1KEYS , 10I4)
             IF (K1 .LT. 7) GO TO 5
             READ 11, M
             READ 434, (SRES(I), I = 1,M)
       434   FORMAT (5D16.7)
       5     IF (K8 .NE. 1) GO TO 8
             READ 20, T LIM
             SLIM = -DCBRT(TLIM**2)
             PRINT 30, TLIM, SLIM
       8     EXPONT = DEXP((TLIM + TLIM) / 3.)
             K6 = K6 + 1
             INSUR = 0
             IF (K2 .LT. 10) GO TO 16
             K2 = K2 - 10
             INSUR = 1
```

```
      16   MPCH = 0
           IF (K5 .LT. 10) GO TO 17
           K5 = K5 - 10
           MPCH = 1
      17   IF (K4 .NE. 1) GO TO 3
           READ 20, (CONTR(I), I = 1,9)
           CONTR(10) = -1000.D0
           READ 4, (J SMBL(I), I = 1,9)
      4    FORMAT (9A1)
      3    READ 10, N, FREQ
      10   FORMAT (I2, D10.1)
           IF (N.EQ.0) GO TO 999
      2    PRINT 12, N, FREQ
      12   FORMAT (I3, 8H LAYERS, ,F10.1, 3H HZ)
           READ 20, (Z(I), I=1,N)
           PRINT30, (Z(I), I=1,N)
           READ 20, (C(I), I = 1,N)
      20   FORMAT (8D10.4)
           PRINT30, (C(I), I = 1,N)
           READ 20, (CB(I), I = 1,N)
           PRINT30, (CB(I), I = 1,N)
           READ 20, (GAMMA(I), I = 1,N)
           PRINT30, (GAMMA(I), I = 1,N)
           READ  20, (DPK(I), I = 1,N)
           PRINT 30, (DPK(I), I = 1,N)
           READ 20, (BLPK(I), I = 1,N)
           PRINT30, (BLPK(I), I = 1,N)
           READ  20, (RHO(I), I = 1,N)
           PRINT 30, (RHO(I), I = 1,N)
           IF (FREQ .GT. 0.) GO TO 18
           FREQ = - FREQ
           ATTEN = 0.
           GO TO 19
      18   F SQ = (FREQ / 1000.)**2
           ATTEN = .1 * F SQ / (1. + F SQ) + 40. * F SQ / (4100. + F SQ)
      19   ATTEN = ATTEN * 1.0936
           PRINT 14,ATTEN
      14   FORMAT (8H ATTEN = ,G10.5, 5HDB/KM )
           ATTEN = ATTEN / 1000.D0
      30   FORMAT (9F14.5)
     C  COMPLETE PROFILE
           DO 33 I = 1,N
           IF (RHO(I) .EQ. 0.) RHO(I) = 1.02
           IF (CB(I) .NE. 0.) GO TO 31
           CB(I) = C(I+1)
      31   IF (C(I) .NE. 0.) GO TO 32
           C(I) = CB(I-1)
      32   IF (DPK(I) .GE. 0.) GO TO 34
           CI(I) = CBI(I-1)
           GO TO 36
      34   CI(I) = 0.
           IF (DPK(I) .EQ. 0.) GO TO 36
           T = 27287.52708 / DPK(I)
           CI(I) = T - SQRT((T - C(I)) * (T + C(I)))
      36   CBI(I) = 0.
           IF (GAMMA(I) .NE. 0.) GO TO 38
     C     **BOTH SOUND SPEEDS GIVEN
```

```
114             IF (BLPK(I) .LE. 0.) GO TO 37
115             T = 27287.52708 / BLPK(I)
116             CBI(I) = T - SQRT ((T - CB(I)) * (T + CB(I)))
117          37 T = C(I) * (C(I)**2 - 3. * CI(I)**2)
118             TI = CI(I) * (3. * C(I)**2 - CI(I)**2)
119             IF (BLPK(I) .LT. 0.) GO TO 39
120             TEMP = CB(I)**2 - CBI(I)**2
121             TEMP1 = 2. * CB(I) * CBI(I)
122             DENOM = TEMP**2 + TEMP1**2
123             TEMP = TEMP / DENOM
124             TEMP1 = -TEMP1 / DENOM
125             GAMMA(I) = 0.5 * (C(I) - (T * TEMP - TI * TEMP1)) /
126            * (Z(I+1) - Z(I))
127             GAMMAI(I) = 0.5 * (CI(I) - (T * TEMP1 + TI * TEMP)) /
128            * (Z(I+1) - Z(I))
129             IF (I .EQ. N) GO TO 27
130             GO TO 33
131          C  **SPECIAL CASE, GRADIENT REAL NUMBER
132          39 IF (CI(I) .EQ. 0.) GO TO 42
133             TEMP = CB(I)**2
134             TEM = TEMP**2
135             TEMP1 = CI(I)
136             COEF1 = CI(I)
137             COEF2 = 2. * TEMP * CI(I) + TI
138             COEF3 = 2. * T * CB(I)
139             COEF4 = TEM * CI(I) - TEMP * TI
140             OLDFN = 1.D20
141             DO 41 J = 1,10
142             FN= (((COEF1 * TEMP1) + COEF2) * TEMP1 + COEF3) * TEMP1 + COEF4
143             FP = ((4. * COEF1 * TEMP1) + 2. * COEF2) * TEMP1 + COEF3
144             TEMP1 = TEMP1 - FN / FP
145             IF (FN .GE. OLDFN) GO TO 43
146             OLDFN = FN
147          41 CONTINUE
148          43 CBI(I) = TEMP1
149             GAMMA(I) = .5 *(.5 * (CI(I) *(TEMP - CBI(I)**2) - TI) /
150            * (CB(I) * TEMP1) + C(I)) / (Z(I+1) - Z(I))
151             GO TO 28
152          42 GAMMA(I) = C(I) - T / (CB(I)**2 * (Z(I+1) - Z(I)) * 2.)
153             GO TO 33
154          C  **SOUND SPEED AND GRADIENT GIVEN
155          38 IF (I .EQ. N) GO TO 33
156             T = C(I) * (C(I)**2 - 3. * CI(I)**2)
157             TI = CI(I) * (3. * C(I)**2 - CI(I)**2)
158             IF (BLPK(I) .EQ. 0.) GO TO 29
159             IF (BLPK(I) .LT. 0.) GO TO 28
160             TEMP = (BLPK(I) / 54575.05416)**2 / (Z(I+1) - Z(I)) * 0.5
161             T = T * TEMP
162             TI = TI * TEMP
163             TEMP = .5 * C(I) / (Z(I+1) - Z(I)) - GAMMA(I) + T
164             T = -(TI - SQRT( TI * TI + T * TEMP)) / T
165             CB(I) = 54575.05416 * T / BLPK(I) / (1. + T * T)
166             CBI(I) = CB(I) / T
167             GO TO 37
168          C  **SPECIAL CASE, GRADIENT REAL NUMBER
169          28 TEMP = C(I) - 2. * GAMMA(I) * (Z(I+1) - Z(I))
170             TEM = TEMP**2 + CI(I)**2
```

```
171           XRE = (T * TEMP + TI * CI(I)) / TEM
172           XIM = -(T * CI(I) - TI * TEMP) / TEM
173           TEM = XRE**2 + XIM**2
174           CB(I) = SQRT((XRE + SQRT(TEM)) * .5)
175           CBI(I) = .5 * XIM / CB(I)
176           GAMMAI(I) = 0.
177           GO TO 33
178        29 TEMP = C(I) - 2. * GAMMA(I) * (Z(I+1) - Z(I))
179           CB(I) = SQRT (T / TEMP)
180           GAMMAI(I) = .5 * (CI(I) - TI / CB(I)**2) / (Z(I+1) - Z(I))
181           GO TO 33
182        27 N = N - 1
183        33 CONTINUE
184      C  COMPUTE USEFULL QUANTITIES
185           PRINT 58
186        58 FORMAT (7X,6H RE M ,8X,6H IM M ,9X,5H L/KM,8X,6H RE C ,8X,
187          * 6H IM C ,5X,12H RE C BOTTOM,4X,12H IM C BOTTOM,10X,9H GRADIENT )
188           OMEGA = 6.28318530700 * FREQ
189           DO 40 I = 1,N
190           TEMP = C(I)**2 + CI(I)**2
191           CAY(I) = OMEGA * C(I) / TEMP
192           CAYI(I) = -OMEGA * CI(I) / TEMP
193           CAY SQ(I) = CAY(I)**2 - CAYI(I)**2
194           CAY SQI(I) = 2.D0 * CAY(I) * CAYI(I)
195           TEMDR = -2. * (GAMMA(I) * CAY SQ(I) - GAMMAI(I) * CAY SQI(I))
196           TEMDI = -2. * (GAMMA(I) * CAY SQI(I) + GAMMAI(I) * CAY SQ(I))
197           G CU(I) = (TEMDR * C(I) + TEMDI * CI(I)) / TEMP
198           G CUI(I) = (TEMDI * C(I) - TEMDR * CI(I)) / TEMP
199           TEM1 = DCBRT(-DSQRT( GAMMA(I)**2 + GAMMAI(I)**2) * 2.*DMEGA**2)
200           TEM1I = DATAN (ABS(GAMMAI(I) / GAMMA(I)))/ 3.
201           CRTG = TEM1 * DCOS(TEM1I)
202           CRTGI = TEM1 * DSIN(TEM1I)
203           IF (GAMMA(I) .LT. 0.) CRTG = -CRTG
204           IF (GAMMAI(I).LT. 0.) CRTGI = -CRTGI
205           G(I) = (C(I) * CRTG + CI(I) * CRTGI) / TEMP
206           GI(I) = (C(I) * CRTGI - CI(I) * CRTG) / TEMP
207           CON(I) = G(I) * C(I) - GI(I) * CI(I)
208           CON(I) = OMEGA**2 / CON(I)**2
209           XMI = -GI(I) * (Z(I+1) - Z(I))
210           XM = -G(I) * (Z(I+1) - Z(I))
211           DPK(I) = -8686.D0 * CAYI(I)
212           PRINT 30, XM, XMI, DPK(I), C(I), CI(I), CB(I), CBI(I)
213          * ,GAMMA(I), GAMMAI(I)
214           G SQI(I) = 2. * G(I) * GI(I)
215        40 G SQ(I) = G(I)**2 - GI(I)**2
216      C  FIND MODES
217           NXTRA=0
218           IJ FLAG=0
219           NN = NN + 1
220           IF (K5 .EQ. 1) GO TO 15
221           DO 50 NN = 1,350
222        15 IF (IJ FLAG .EQ. 1) GO TO 53
223        52 IF (NXTRA .GT. 0) GO TO 44
224           READ 60, V,VI,STEP,STEPI,NXTRA
225        60 FORMAT (4D10.4,I10)
226           IF (NXTRA .GE.0) GO TO 62
227           V = V + VI * 1.D-10
```

```
228            VI = STEP + STEPI * 1.D-10
229            GO TO 85
230         62 IF (V) 142,301,70
231        142 IF(STEP) 44,44,143
232          C  SEARCH FOR MODE
233        143 SIZE3 = -1.
234            SIZE2=0
235            IJ FLAG=1
236            V=-V
237            IF (NXTRA) 55,55,54
238         55 NXTRA = 20
239         54 XTRA = NXTRA
240            HOP = (STEP - V) / XTRA
241            HOPI=0.
242            IF(STEPI.NE.0.) HOPI=(STEPI-VI)/XTRA
243            DO 47 IJ = 1,NXTRA
244            CALL SETUP
245            DET = VEL
246            DETI = VELI
247            CALL DETNT(N,VEL,VELI)
248            DELTA = VEL
249            DELTI = VELI
250            SIZE = DELTA*DELTA + DELTI*DELTI
251            PRINT 56, V, VI, SIZE, VEL, VELI
252         56 FORMAT (2F12.3, 3D17.5)
253            IF ((SIZE2.LT.SIZE3).AND.(SIZE.GT.SIZE2)) GO TO 45
254           46 SIZE3=SIZE2
255            SIZE2=SIZE
256            V = V + HOP
257            VI=VI+HOPI
258            GO TO 47
259         45 V = V - HOP
260            TEMP = HOP / (SIZE - SIZE2)
261            DELTI = TEMP * (DET * VELI - DETI * VEL)
262            TEMP = .5D0 * (SIZE3 - SIZE) / (SIZE3 + SIZE - SIZE2 - SIZE2)
263            DELTA = HOP * TEMP
264            IF(HOPI.EQ.0) GO TO 49
265            VI=VI-HOPI
266            DELTAI=HOPI*TEMP
267            GO TO 49
268         47 CONTINUE
269            IJ FLAG=0
270            NXTRA=0
271            GO TO 52
272           53 SIZE 2=-1.
273            SIZE=0
274            GO TO 46
275         44 NXTRA = NXTRA - 1
276            V = 3. * (PHASE V(NN-1) - PHASE V(NN-2)) + PHASE V(NN-3)
277            VI = 3.* (PHASI V(NN-1) - PHASI V(NN-2)) + PHASI V(NN-3)
278            STEP = (PHASE V(NN-1) - PHASE V(NN-2)) * .0001
279         70 CALL SETUP
280            CALL DETNT(N,DET,DETI)
281         80 FORMAT (/, 2D20.11, 4D13.4)
282            VEL = DET
283            VELI = DETI
284            DELTA = STEP
```

```
285            DELTI = STEPI
286            IF (DELTA .NE.0.) GO TO 49
287            IF (DELTI .EQ.0.) DELTA = .01
288       49   SIZE2 = 100.
289            RX = DET**2 + DETI**2
290            IF (K6 .LT. 3) PRINT 80, V, VI, DET, DETI, SIZE, CNTR
291            J = 0
292       48   J = J + 1
293            IF (J .GT. 15) GO TO 51
294            V = V + DELTA
295            VI = VI + DELTI
296            IF (VI) 83,84,85
297       83   DELTI = DELTI - VI
298       84   VI = 1.D-18
299       85   CALL SETUP
300            NNN = N + N - 1
301            DO 82 IA = 1,NNN
302            DO 82 IB = 1,4
303            BI(IA,IB) = Q(IA,IB)
304       82   B(IA,IB) = A(IA,IB)
305            CALL DETNT(N,DET,DETI)
306            IF (K6 .NE. 1) GO TO 72
307       71   PRINT 81, V, VI, DET, DETI, SIZE, CNTR
308       81   FORMAT (2D20.11, 4D13.4)
309       72   IF (NXTRA .LT. 0) GO TO 51
310            TEMNR =  DET * DELTA - DETI * DELTI
311            TEMNI = DETI * DELTA + DET * DELTI
312            TEMDR = VEL - DET
313            TEMDI = VELI - DETI
314            TEMDEN = TEMDR*TEMDR + TEMDI*TEMDI
315            IF (TEMDEN .EQ. 0.) GO TO 51
316            TEMRNU = TEMNR*TEMDR + TEMNI*TEMDI
317            TEMINU = TEMNI*TEMDR - TEMNR*TEMDI
318            DELTA = TEMRNU/TEMDEN
319            DELTI = TEMINU/TEMDEN
320      C * * * THE NEXT CONSTANT DEPENDS ON WORD LENGTH AND SIZE OF PHASE VELOCITY * *
321            IF (ABS(DELTA) .LT. 1.D-14) GO TO 51
322            SIZE = DELTA*DELTA + DELTI*DELTI
323            IF ((SIZE.GT.SIZE2).AND.(J.GT.3)) GO TO 51
324       92   SIZE2 = SIZE * 2.
325            VEL = DET
326            VELI = DETI
327            GO TO 48
328      C  FIND DEPTH FUNCTIONS
329       51   IF (INSUR .EQ. 0) GO TO 61
330            TRE = (DET**2 + DETI**2) / RX
331            IF (TRE .LT. 1E-10) GO TO 61
332            PRINT 998, NN, TRE
333       998  FORMAT (5H MODE ,I4,23H FAILED TO CONVERGE --  , E9.2)
334            GO TO 999
335       61   IF (MPCH .EQ. 0) GO TO 63
336            IF (NXTRA .LT. 0) GO TO 63
337            TEM1 = V * 1.D4
338            COL(1) = TEM1
339            TEMP = COL(1)
340            COL(2) = (TEM1 - TEMP) * 1.D10
341            TEM1 = VI * 1.D4
```

```
342             COL(3) = TEM1
343             TEMP = COL(3)
344             COL(4) = (TEM1 - TEMP) * 1.D10
345             COL(5) = -NN
346             PUNCH 64, (COL(I), I = 1,5)
347          64 FORMAT (5I10)
348          63 AF(1,NN) = A(1,3)
349             AG(1,NN) = Q(1,3)
350             BF(1,NN) =-A(1,4)
351             BG(1,NN) =-Q(1,4)
352             PHASE V(NN) = V
353             PHASI V(NN) = VI
354             IF (K6 .EQ. 1) GO TO 73
355          74 PRINT 81, V, VI, DET, DETI, SIZE, CNTR
356          73 LL = N - 1
357             IF (LL-1) 95,96,97
358          96 I = 0
359             GO TO 98
360          97 DO 110 J = 2,LL
361             I = J + J - 2
362             TEMNR = A(I,2)*AF( J-1 ,NN) - Q(I,2)*AG( J-1 ,NN)
363             TEMNI = Q(I,2)*AF( J-1 ,NN) + A(I,2)*AG( J-1 ,NN)
364             TEMDR = A(I,3)*A( I+1 ,4) - Q(I,3)*Q( I+1 ,4) -
365            1    A(I,4)*A( I+1 ,3) + Q(I,4)*Q( I+1 ,3)
366             TEMDI = Q(I,3)*A( I+1 ,4) + A(I,3)*Q( I+1 ,4)  -
367            1     Q(I,4)*A( I+1 ,3) - A(I,4)*Q( I+1 ,3)
368             TEMDEN = TEMDR*TEMDR + TEMDI*TEMDI
369             TEMRNU = TEMNR*TEMDR + TEMNI*TEMDI
370             TEMINU = TEMNI*TEMDR - TEMNR*TEMDI
371             TEMP = TEMRNU / TEMDEN
372             TEMPI = TEMINU / TEMDEN
373             BF(J,NN) = -(TEMP*A( I+1 ,4) - TEMPI*Q( I+1 ,4))
374             BG(J,NN) = -(TEMPI*A( I+1 ,4) + TEMP*Q( I+1 ,4))
375             AG(J,NN) = TEMPI*A( I+1 ,3) + TEMP*Q( I+1 ,3)
376         110 AF(J,NN) = TEMP*A( I+1 ,3) - TEMPI*Q( I+1 ,3)
377          98 TEMNR = -(A(I+2,2) * AF(LL,NN) -Q(I+2,2) * AG(LL,NN))
378             TEMNI = -(Q( I+2 ,2)*AF(LL,NN) + A( I+2 ,2)*AG(LL,NN))
379             TEMDEN = A( I+2 ,3)*A( I+2 ,3) + Q( I+2 ,3)*Q( I+2 ,3)
380             TEMRNU = TEMNR*A( I+2 ,3) + TEMNI*Q( I+2 ,3)
381             TEMINU = TEMNI*A( I+2 ,3) - TEMNR*Q( I+2 ,3)
382             BF(N,NN) = TEMRNU / TEMDEN
383             BG(N,NN) = TEMINU / TEMDEN
384          95 AF(N,NN) = 0.
385             AG(N,NN) = 0.
386          C  FIND NORMALIZING FACTOR
387             D(NN) = 2.12429296D0 * RHO(1)**3 / G(1)
388             DI(NN) = 0.
389             DO 111 I = 2,N
390             TEMRSP = AF( I-1 ,NN)*B( 2*I-2 ,2) - AG( I-1 ,NN)*BI( 2*I-2 ,2)  +
391            1    BF( I-1 ,NN)*B( 2*I-2 ,1) - BG( I-1 ,NN)*BI( 2*I-2 ,1)
392             TEMISP = AG( I-1 ,NN)*B( 2*I-2 ,2) + AF( I-1 ,NN)*BI( 2*I-2 ,2)  +
393            1    BG( I-1 ,NN)*B( 2*I-2 ,1) + BF( I-1 ,NN)*BI( 2*I-2 ,1)
394             AX1 = TEMRSP*TEMRSP - TEMISP*TEMISP
395             AX1I = TEMRSP * TEMISP
396             AX1I = AX1I + AX1I
397             TEMDR = (G(I-1)**2 + GI(I-1)**2)
398             TEMDI = G(I)**2 + GI(I)**2
```

```
399            TEMP = (RHO(I-1) / RHO(I)) / TEMDI
400            TEM1  =(ZB (I-1) * G(I-1) + ZBI(I-1) * GI(I-1)) / TEMDR
401           * -(ZT (I) * G(I) + ZTI(I) * GI(I)) * TEMP
402            TEM1I =(ZBI(I-1) * G(I-1) - ZB (I-1) * GI(I-1)) / TEMDR
403           * -(ZTI(I) * G(I) - ZT (I) * GI(I)) * TEMP
404            TEMRSP = AF( I-1 ,NN)*B( 2*I-1 ,2) - AG( I-1 ,NN)*BI( 2*I-1 ,2)  +
405           1    BF( I-1 ,NN)*B( 2*I-1 ,1) - BG( I-1 ,NN)*BI( 2*I-1 ,1)
406            TEMISP = AG( I-1 ,NN)*B( 2*I-1 ,2) + AF( I-1 ,NN)*BI( 2*I-1 ,2)  +
407           1    BG( I-1 ,NN)*B( 2*I-1 ,1) + BF( I-1 ,NN)*BI( 2*I-1 ,1)
408            AX2 = TEMRSP*TEMRSP - TEMISP*TEMISP
409            AX2I = TEMRSP * TEMISP
410            AX2I = AX2I + AX2I
411            TEMDR = RHO(I-1) / (G CU(I-1)**2 + G CUI(I-1)**2)
412            TEMDI = RHO(I  ) / (G CU(I  )**2 + G CUI(I  )**2)
413            TEM2 = G CU(I-1) * TEMDR - G CU(I) * TEMDI
414            TEM2I = G CUI(I) * TEMDI - G CUI(I-1) * TEMDR
415            TEMR1 = AX1*TEM1 - AX1I*TEM1I
416            TEMI1 = AX1I*TEM1 + AX1*TEM1I
417            TEMR2 = AX2 * TEM2 - AX2I * TEM2I
418            TEMI2 = AX2I * TEM2 + AX2 * TEM2I
419            D(NN) = D(NN) + TEMR1 / RHO(I-1) + TEMR2
420            DI(NN) = DI(NN) + TEMI1 / RHO(I-1) + TEMI2
421       111  CONTINUE
422            IF ( K1 .GT. 3) DA(NN) = DSQRT((D(NN)**2 + DI(NN)**2) * FREQ /
423           *  PHASE V(NN))
424            EIGEN(NN) = LAMBDA
425            EIGENI(NN) = LAMBDI
426            IF (K6 .GT. 2) GO TO 131
427            L = 0
428            K = 24
429            DO 112 I = 1,N
430            L = L + 1
431            COL(L) = SNGL(ZT(I)) * 100.
432            L = L + 1
433            COL(L) = SNGL(ZTI(I)) * 1000.
434            K = K + 1
435            COL(K) = SNGL(ZB(I)) * 100.
436            K = K + 1
437            COL(K) = SNGL(ZBI(I)) * 1000.
438       112  CONTINUE
439            PRINT 130, (COL(I), I=1,L)
440            PRINT 130, (COL(I), I=25,K)
441       130  FORMAT (4H Z = , 11(I6,I5))
442            M = N + N
443            PRINT 132, (RATIO(I), I = 1,M)
444       132  FORMAT (11(1X,2F5.3))
445       131  DB LOSS(NN) = - LAMBDI * 8686.D0
446            PHINV = V * PHASE V(NN-1) /((V - PHASE V(NN-1))* FREQ)
447            PRINT 109, NN,EIGEN(NN),EIGENI(NN),D(NN),DI(NN),PHINV,DB LOSS(NN)
448       109  FORMAT (3H N=,I5,10H  LAMBDA =,2E15.7,4H D= ,2E15.7,
449           * 12H INT RANGE = ,F8.0, 6H L/K =, F8.5)
450            IF (K3 .EQ. 0) GO TO 50
451            CALL RCOEF (K3)
452       50   CONTINUE
453      C  READ IN SOURCE AND RECEIVERS DEPTHS
454        301  NRT = NR
455            NR = 0
```

```
456            NN = NN - 1
457            K1P1 = K1 + 1
458            IF (K1 .NE. 3) GO TO 321
459            NR = NRT
460            GO TO 501
461       321  READ 20, SOURCE
462       320  NR = NR + 1
463            READ 20, RECVRS(NR), FINAL, STEPP
464            IF (NR.GT.50) GO TO 300
465       350  IF (RECVRS(NR) .EQ.0.) GO TO 300
466       310  IF (FINAL .EQ.0.) GO TO 320
467       330  RECVRS(NR+1) = RECVRS(NR) + STEPP
468            IF (RECVRS(NR+1) .GT. FINAL) GO TO 320
469       340  NR = NR + 1
470            IF(NR .GT. 50) GO TO 300
471            GO TO 330
472       300  PRINT 303
473       303  FORMAT (/21H SOURCE AND RECEIVERS )
474            PRINT 21,(DEPTH(I), I = 1,NR)
475       21   FORMAT (8F10.2)
476      C  COMPUTE DEPTH FUNCTIONS
477            DO 500 I = 1,NN
478            LOC = 1
479            DO 305 J = 1,NR
480            IF ((J .EQ. 1) .AND. (K1 .GT. 5)) GO TO 305
481            LCTR = 0
482       380  IF((DEPTH(J) .GE. Z(LOC)).AND.(DEPTH(J) .LT. Z(LOC+1)))GO TO 360
483       371  IF (LOC .GE. N) GO TO 385
484       370  LOC = LOC + 1
485            GO TO 380
486       385  IF (DEPTH(J) .GE. Z(LOC)) GO TO 360
487        390 LOC=1
488            LCTR=LCTR+1
489            IF (LCTR .GT. 2) GO TO 305
490            GO TO 380
491       360  X1 = CAY (LOC) - EIGEN (I)
492            X2 = CAY (LOC) + EIGEN (I)
493            X3 = CAYI(LOC) - EIGENI(I)
494            X4 = CAYI(LOC) + EIGENI(I)
495            TEMP = X1 * X2 - X3 * X4
496            TEMPI = X1 * X4 + X3 * X2
497            TEMDEN = G SQ(LOC) **2 + G SQI(LOC)**2
498            ZE = (TEMP * GSQ(LOC) + TEMPI * G SQI(LOC)) / TEMDEN
499            ZEI = (TEMPI * GSQ(LOC) + TEMP * GSQI(LOC)) / TEMDEN
500            TEM1 = ZE
501            IF (ZE .GT. -7.5) GO TO 438
502            S = CAY(LOC)
503            T = CAYI(LOC)
504            DO 437 K = 1,20
505            TEMP = S**2 + T**2
506            TEMPI = (EIGENI(I) * S - EIGEN(I) * T) / TEMP
507            TEMP = (EIGEN(I) * S + EIGENI(I) * T) / TEMP
508            ZE = ((1. + TEMP) * (1. - TEMP) + TEMPI**2) * CON(LOC)
509            ZEI = -2. * TEMPI * TEMP * CON(LOC)
510            ZR = ZE / -7.5
511            IF (DABS(ZR-1.) .LT. 1.0-3) GO TO 438
512            S = EIGEN(I) + (S - EIGEN(I)) / ZR
```

```
513       437  CONTINUE
514       438  IF (G(LOC) .LT. 0.) GO TO 439
515            ZE = G(LOC) * (DEPTH(J) - Z(LOC)) + TEM1
516            IF (ZE .GT. -7.5) GO TO 442
517            F(J) = 1.D-12
518            FI(J) = 0.
519            GO TO 305
520       439  ZE = G(LOC) * (DEPTH(J) - Z(LOC)) + ZE
521            IF (ZE .GT. SLIM) GO TO 442
522            F(J) = 1.D-12
523            FI(J) = 0.
524            GO TO 305
525       442  ZEI =GI(LOC) * (DEPTH(J) - Z(LOC)) + ZEI
526       302  CALL HANKEL(ZE,ZEI,1)
527            F(J) =(AF(LOC,I)*H1R - AG(LOC,I)*H1I + BF(LOC,I)*H2R - BG(LOC,I)
528           1     *H2I) * RHO(LOC)
529            FI(J) =(AG(LOC,I)*H1R + AF(LOC,I)*H1I + BG(LOC,I)*H2R + BF(LOC,I)
530           1    *H2I) * RHO(LOC)
531       305  CONTINUE
532            IF (K1 .EQ. 2) GO TO 451
533            GO TO 432
534       451  PRINT 270, DEPTH(NR)
535       270  FORMAT(7H1DEPTH ,F5.1,6X,3HE-8,17X,3HE-6,17X,3HE-4,17X,3HE-2,
536           *      17X,3HE 0    )
537       432  IF (K1 .LT. 4) GO TO 431
538            IF (K1 .GT. 5) GO TO 433
539            SRES(I) = (F(1)**2 + FI(1)**2) / DA(I)
540            GO TO 500
541        431 TEMDEN = D(I)*D(I) + DI(I)*DI(I)
542            TEMRE = F(1)*D(I) + FI(1)*DI(I)
543            FD = TEMRE/TEMDEN
544            FDI = (D(I) * FI(1) - DI(I) * F(1)) / TEMDEN
545       433  DO 400 K = 2,NR
546            J = K - 1
547            L = J * NN - NN + I
548            IF (K1 .LT. 6) GO TO 435
549            FF = SRES(I) * (F(K)**2 + FI(K)**2) / DA(I)
550            GO TO 436
551       435  FF = FD * F (K) - FDI * FI(K)
552            FFI = FD * FI(K) + FDI * F(K)
553       436  UU(L) = FF
554            UUI(L) = FFI
555       452  GO TO (400,410,420,400,400,400,400,400), K1P1
556      C  PLOT DEPTH FUNCTIONS
557        420  DO 210 II = 1,120
558        210  COL(II)= 1H
559             DO 220 II= 20,100,20
560        220  COL(II)= 1HI
561             FE = FF * FF + FFI * FFI
562             IF ((FE.GT.1E-20).AND.(FE.LT.10000.)) GO TO 240
563             GO TO 250
564        240  INT = 100.D0 + 2.17147D0 * DLOG(FE)
565             COL(INT) = 1H*
566             GO TO 225
567        250  COL(2) = 1H*
568        225  PRINT 260, COL
569        260  FORMAT (120A1)
```

```
570           GO TO 400
571       C  PRINT DEPTH FUNCTIONS
572        410  F MAG(J) = SQRT (FF * FF + FFI * FFI)
573             IF (FF) 430,440,450
574        430  F ANG(J) = ATAN(FFI / FF) * 57.29577951D0 + 180.D0
575             GO TO 400
576        440  F ANG(J) = 90.
577             GO TO 400
578        450  F ANG(J) = ATAN(FFI / FF) * 57.29577951D0
579        170  FORMAT (  10F12.4)
580        180  FORMAT(/10E12.3)
581        400  CONTINUE
582             IF (K1.EQ.1) GO TO 441
583             GO TO 500
584        441  PRINT 180, (F MAG(K), K = 1,J)
585             PRINT 170, (F ANG(K), K = 1,J)
586        500  CONTINUE
587       C  CALCULATE ATTENUATION AND READ IN RANGES
588             IF ((K1 .EQ. 4).OR.(K1 .EQ. 5)) PRINT 180, (SRES(K), K= 1,NN)
589             IF (K1 .EQ. 5) PUNCH 434, (SRES(K), K = 1,NN)
590             NR=NR-1
591             IF (K2  .LT. 3) GO TO 501
592             IF (K2 .EQ. 4) K8 = 3
593             IF (K2 .EQ. 3) K8 = 2
594             K2 = 0
595        501  KX = K2 + 1
596             GO TO (561,551,551), KX
597        551  PRINT 533, NN,N, C(1), Z(2), C(2), Z(3), C(3), Z(4), C(4),
598            *  SOURCE, RECVRS(40), FREQ
599        533  FORMAT (1H1, 2I5, 10F10.4)
600             ICTR=0
601             R LOS(1) = 120.
602             LEVEL(1) = 1
603             DO 562 I = 1,5
604             P LEV(I)=40.
605             J COU(I)=4
606             J COUNT(I)=-6
607        562  CONTINUE
608             IF((K2 .EQ. 2).AND.(NR .GT. 5))GO TO 772
609             GO TO 561
610        772  NR = 5
611        561  NL = NN
612             PRINT 524, NL
613        524  FORMAT (I8, 13H MODES IN SUM  )
614             LL = 1
615             IF (K9 .GT. 0) NL = K9
616             READ 20, RANGE, FINAL R, STEP R
617             IF (K8 .EQ. 3) PUNCH 30, RANGE, FINAL R, STEP R
618             IF (RANGE) 563,1,564
619        563  NN = NN + 1
620             READ 11, K1, K2, K3, K4, K5, K6, K7, K8, K9
621             PRINT 11, K1, K2, K3, K4, K5, K6, K7, K8, K9
622             GO TO 301
623        564  IF (FINALR .LE. 0.) GO TO 550
624             FINAL R = FINAL R + 1.D-3
625        560  IF (RANGE .GE. FINALR) GO TO 561
626        550  R ATTEN = RANGE * ATTEN - 9.9429946
```

```
627             IF (K1 .GT. 5) RATTEN = 0.D0
628             IF (K1 .GT. 5) GO TO 536
629             IF (K7 .EQ. 2) RATTEN = 4.3429448D0 * DLOG(FREQ)
630             IF (RANGE * DB LOSS(NL) .LT. 15.D4) GO TO 522
631       521   NL = NL - 1
632       522   DO 520 I = 1,NL
633             IF (K7 .LT. 2) GO TO 523
634             FMAG(I) = PHASE V(I)
635             GO TO 520
636       523   TEMP RE = EIGEN(I) * RANGE
637             TEMPIM = EIGENI(I)*RANGE
638             CALL HZERO(TEMPRE,TEMPIM)
639             HZERO2(I) = HZEROR
640             IF (K7 .EQ. 0) GO TO 520
641             FMAG(I) = HZEROR**2 + HZEROI**2
642       520   HZER2I(I) = HZEROI
643       536   L = 0
644             DO 540 J = 1,NR
645             FF = 0
646             FFI = 0
647             TEMP = 0.D0
648             DO 530 I = 1,NL
649             K = L + I
650             IF (K1 .LT. 6) GO TO 537
651             TEMP = TEMP + UU(K)
652             GO TO 530
653       537   IF (K7 .EQ. 0) GO TO 534
654             TEMP = TEMP + FMAG(I) * (UUI(K)**2 + UU(K)**2)
655             GO TO 530
656       534   TEMIM = UUI(K) * HZERO2(I) + UU(K) * HZER2I(I)
657             TEMRE = UU(K) * HZERO2(I) - UUI(K) * HZER2I(I)
658             FF = FF + TEMRE
659             FFI = FFI + TEMIM
660       530   CONTINUE
661             IF (K1 .GT. 5) GO TO 535
662             IF (K7 .GT. 0) GO TO 535
663             TEMP = FF**2 + FFI**2
664       535   T RE = TEMP
665             RX = -4.3429448 * ALOG(T RE) + R ATTEN
666             R LOSS(J) = RX
667             IF (K4 .LT. 2) GO TO 545
668             T RE = -4.3429448 * ALOG(UU(K)**2 + UUI(K)**2)
669             PRINT 170, RECVRS(J), RX, T RE
670             IF (K4 .NE. 3) GO TO 545
671       545   CONTINUE
672             L = L + NN
673             IF (K8 .LT. 2) GO TO 540
674             IF (K8 .EQ. 3) GO TO 538
675             LPCH = -RLOSS(J) * 10.D0 + 1400.5D0
676             IF (LPCH .LT. 0) LPCH = 0
677             IF (LPCH .GT. 999) LPCH = 999
678             LOSPCH(J,LL) = LPCH
679             IRNG = RANGE / 1000.D0
680             IF (LL .EQ. 25) PUNCH 903, IRNG, (LOSPCH(J,LLL),LLL = 1,25)
681       903   FORMAT (I5,25I3)
682             GO TO 540
683       538   CONTINUE
```

```
684           LOSS(J) = (140.05 - RX) * 10.
685           IF (LOSS(J) .LT. 0) LOSS(J) = 0
686           IF (LOSS(J) .GT. 999) LOSS(J) = 999
687       540 CONTINUE
688           GO TO (770,780,716),KX
689       C  PLOT DB LOSSES
690         712 COL(15)=1HI
691           COL(39)=1HI
692           COL(63)=1HI
693           COL(87)=1HI
694           COL(111)=1HI
695           COL(27)=1HX
696           COL(51)=1HX
697           COL(75)=1HX
698           COL(99)=1HX
699           I PLACE = 135
700           DO 787  J = 1,NR
701           I PLACE = I PLACE - 24
702       783 IPLOT = P LEV(J) -  R LOSS(J)
703           IF (I PLOT .GT. 10) GO TO 776
704           GO TO 777
705       776 P LEV(J) = P LEV(J) - 20.
706           J COUNT(J) = J COUNT(J) - 2
707           J COU(J)=J COU(J)-2
708       786 COL(I PLACE + 1) = 1H0
709           IF (P LEV(J) - 100.) 778,779,781
710       778 JC = J COU(J) + 1
711           COL(IPLACE) = ING(JC)
712           GO TO 783
713       779 COL(I PLACE) = 1H0
714           GO TO 782
715       781 JC = J COUNT(J) + 1
716           COL(IPLACE) = ING(JC)
717       782 COL(I PLACE - 1) = 1H1
718           GO TO 783
719       777 IF (I PLOT  .LT. -9) GO TO 784
720           GO TO 785
721       784 P LEV(J) = P LEV(J) + 20.
722           J COU(J)=J COU(J)+2
723           J COUNT(J) = J COUNT(J) + 2
724           GO TO 786
725       785 IPP = I PLACE + IPLOT
726           COL(IPP) = 1H+
727       787 CONTINUE
728           GO TO 750
729       C  CONTOUR LOSS FIELD
730       780 DO 590 JJ = 1,120
731       590 COL(JJ) = 1H
732           COL(31)=1HI
733           COL(61)=1HI
734           COL(91)=1HI
735           DO 640 JJ = 2,41
736           LEV = 1
737       620 IF (RLOS (JJ) .LT. CONTR(LEV)) GO TO 600
738           GO TO 610
739        600 LEV=LEV+1
740           GO TO 620
```

```
741       610  IF (LEV .EQ. LEVEL(JJ)) GO TO 640
742       650  IF (LEV .GT. LEVEL(JJ)) GO TO 660
743            GO TO 670
744       660  II = LEVEL(JJ)
745            GO TO 680
746       670  II = LEV
747       680  JJJ = 124 - 3*JJ
748            COL(JJJ) = J SM8L(II)
749            LEVEL(JJ) = LEV
750       640  CONTINUE
751            COL(1) = 1HI
752             PRINT 261, (COL(I1), I1 = 1,119)
753       716  DO 690 JJ = 1,120
754       690  COL(JJ) = 1H
755            ICTR = ICTR + 1
756            IF (ICTR .EQ. 10) GO TO 700
757            GO TO 714
758       700  TEMP = (RANGE + 1.) / 10000.
759            IND = TEMP
760            COL(2) = ING(IND+1)
761            TEMP1 = IND
762            TEMP = (TEMP - TEMP1) * 10.
763            IND = TEMP
764            COL(3) = ING(IND+1)
765            TEMP1 = IND
766            IND = (TEMP - TEMP1) * 10.
767            COL(5) = ING(IND+1)
768            COL(4)=1H.
769            COL(6)=1HK
770            COL(7)=1HY
771            COL(8)=1HD
772            COL(9) = 1HS
773            ICTR=0
774       714  GO TO (710,712),K2
775        710 COL(31)=1HI
776            COL(61)=1HI
777            COL(91)=1HI
778            DO 720 JJ = 1,40
779            TEMP = LEVEL(JJ)
780            TEMPI = 0.
781       830  IF (LEVEL(JJ) .GT. LEVEL(JJ+1)) GO TO 730
782            GO TO 740
783       730  II = LEVEL(JJ) - 1
784            KK = 1
785       860  EX = (CONTR(II) - R LOS (JJ) )/ (RLOS (JJ+1) - CONTR(II))
786            DO 760 LL = 1,3
787            IF (EX .LT. TEST(LL)) GO TO 800
788       760  CONTINUE
789            LL = 4
790       800  JJLL = 125 - 3*JJ - LL
791            COL(JJLL) = J SM8L(II)
792            GO TO (810,820),KK
793       810  LEVEL(JJ) = LEVEL(JJ) - 1
794            GO TO 830
795       740  IF (LEVEL(JJ) .LT. LEVEL(JJ+1)) GO TO 840
796            GO TO 720
797       840  II = LEVEL(JJ)
```

```
798             KK=2
799             GO TO 860
800          820  LEVEL(JJ) = LEVEL(JJ) + 1
801             GO TO 740
802          720  LEVEL(JJ) = TEMP
803             COL(1) = 1HI
804          750   PRINT 261, (COL(I1), I1 = 1,119)
805          261  FORMAT (1X, 119A1)
806             GO TO 581
807        C  PRINT DB LOSSES
808          770  PRINT 580, RANGE, (R LOSS(K), K = 1,NR)
809             LL = LL + 1
810             IF (LL .GT. 25) LL = 1
811          580  FORMAT (F9.0, 2X, 18F6.1)
812          581  RANGE = RANGE + STEP R
813             IF (K8 .NE. 3) GO TO 560
814             PUNCH 980, (LOSS(I), I= 1,NR)
815          980  FORMAT (26I3)
816             GO TO 560
817          999  STOP
818             END
```

```
      SUBROUTINE SETUP
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DOUBLE PRECISION  LAMBDA,LAMBDI
      COMMON /HAN/ H2R,H2I,H1R,H1I,H2PR,H2PI,H1PR,H1PI,EXPONT
      COMMON /EXPO/ EXSUM, CNTR, RATIO(25)
      COMMON/DETMNT/ A(25,4),Q(25,4)
      COMMON/INPUT/ Z(12), N, OMEGA, V, VI, CON(12), GSQ(12),
     1 CAY(12), LAMBDA, LAMBDI, G(12)
     2,RHO(12), GI(12), G SQI(12), CAYI(12)
       COMMON /LIMIT/ TLIM, EXPON , SLIM
      COMMON/PARTS/ ZT(12),ZTI(12),ZB(12),ZBI(12)
      DENOM = V * V + VI * VI
      LAMBDA = OMEGA * V / DENOM
      LAMBDI = -OMEGA * VI/ DENOM
      M = N - 1
      DO 10 I = 0,M
      IF (I .EQ. 0) GO TO 35
      IF (ZR .GT. -7.4) GO TO 25
      IF (G(I) .LT. 0.) GO TO 25
      ZE = G(I) * (Z(I+1) - Z(I)) + ZE
      IF (ZE .LT. -7.5) ZE = -7.5
      GO TO 26
   25 CONTINUE
      ZE = G(I) * (Z(I+1) - Z(I)) + ZR
      IF (ZE .LT. SLIM) ZE = SLIM
   26 CONTINUE
      ZQ = GI(I) * (Z(I+1) - Z(I)) + ZI
   30 ZB(I) = ZE
      ZBI(I) = ZQ
      CALL HANKEL(ZE,ZQ,0)
      ZB(I) = ZE
      ZBI(I) = ZQ
      RATIO(2*I) = EXPONT
      A(2*I,1) = H2R * RHO(I)
      Q(2*I,1) = H2I * RHO(I)
      A(2*I,2) = H1R * RHO(I)
      Q(2*I,2) = H1I * RHO(I)
      A(2*I+1,1) = H2PR * G(I) - H2PI * GI(I)
      Q(2*I+1,1) = H2PI * G(I) + H2PR * GI(I)
      A(2*I+1,2) = H1PR * G(I) - H1PI * GI(I)
      Q(2*I+1,2) = H1PI * G(I) + H1PR * GI(I)
   35 CONTINUE
      GSABS = G SQ(I+1)**2 + G SQI(I+1)**2
      X1 = CAY(I+1) - LAMBDA
      X2 = CAY(I+1) + LAMBDA
      X3 =CAYI(I+1) - LAMBDI
      X4 =CAYI(I+1) + LAMBDI
      X = X1 * X2 - X3 * X4
      Y = X1 * X4 + X3 * X2
      ZT(I+1) = (X * G SQ(I+1) + Y * G SQI(I+1)) / GSABS
      ZTI(I+1) = (Y * G SQ(I+1) - X * G SQI(I+1)) / GSABS
      ZR = ZT(I+1)
      ZI = ZTI(I+1)

      ZE = ZR
      ZQ = ZI
```

```
      IF (ZR .GT. -7.5) GO TO 40
      S = CAY(I+1)
      T = CAYI(I+1)
      CON =          (G(I+1) * S + GI(I+1) * T) / (S**2 + T**2)
      CON = 1. / CON**2
      DO 36 J = 1,20
      TEMP = S**2 + T**2
      TEMPI = (LAMBDI * S - LAMBDA * T) / TEMP
      TEMP = (LAMBDA * S + LAMBDI * T) / TEMP
      ZR = ((1. + TEMP) * ( 1. - TEMP) + TEMPI**2) * CON(I+1)
      R = ZR / -7.5
      IF (DABS(R-1.) .LT. 1.D-3) GO TO 41
      S = LAMBDA + (S - LAMBDA) / R
   36 CONTINUE
   41 ZI = -2. * TEMPI * TEMP * CON(I+1)
      ZT(I+1) = ZR
      ZTI(I+1) = ZI

   40 CONTINUE
      CALL HANKEL(ZR,ZI,0)
      ZT(I+1) = ZR
      ZTI(I+1) = ZI
      RATIO(2*I+1) = EXPONT
      IF (I .NE. 0) GO TO 45
      A(1,3) = H2R * RHO(1)
      Q(1,3) = H2I * RHO(1)
      A(1,4) = H1R * RHO(1)
      Q(1,4) = H1I * RHO(1)
      GO TO 10
   45 CONTINUE
      A(2*I,3) =-H2R * RHO(I+1)
      Q(2*I,3) =-H2I * RHO(I+1)
      A(2*I,4) =-H1R * RHO(I+1)
      Q(2*I,4) =-H1I * RHO(I+1)
      A(2*I+1,3) =-H2PR * G(I+1) + H2PI * GI(I+1)
      Q(2*I+1,3) =-H2PI * G(I+1) - H2PR * GI(I+1)
      A(2*I+1,4) =-H1PR * G(I+1) + H1PI * GI(I+1)
      Q(2*I+1,4) =-H1PI * G(I+1) - H1PR * GI(I+1)
   10 CONTINUE
      A( 2*N-2 ,4) = 0.
      Q( 2*N-2 ,4) = 0.
      A( 2*N-1 ,4) = 0.
      RETURN
      END
```

```
      SUBROUTINE DETNT(N,DET,DETI)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DOUBLE PRECISION DET, DETI
      COMMON /EXPO/ EXSUM, CNTR, RATIO(25)
      COMMON/DETMNT/ A(25,4), Q(25,4)
      DLOSS = 1.
      CNTR = 0.
      DET = A(1,3)
      DETI = Q(1,3)
      LIM = N + N - 3
      DO 100 I=1,LIM,2
      J = I
      K = 4
      L = I
      M = 3
      II = 1
      GO TO 500
   10 J = I + 1
      K = 2
      L = J
      M = 1
      GO TO 600
   30 J = I + 2
      L = J
      II = 2
      GO TO 600
   40 L = I + 1
      M = 2
      GO TO 500
   50 K = 3
      M = 3
      II = 3
      GO TO 600
   60 K = 4
      IF (I .EQ. LIM) GO TO 70
      M = 4
      II = 4
      GO TO 600
   70 K = 3
      II = 2
      GO TO 700
  500 C = A(L,M)*A(L,M) + Q(L,M)*Q(L,M)
   80 B = (A(J,K)*A(L,M) + Q(J,K)*Q(L,M)) / C
      BI = (Q(J,K)*A(L,M) - A(J,K)*Q(L,M))/C
      GO TO (10,50), II
  600 TD = A(J,K) - (A(L,M)*B - Q(L,M)*BI)
      TDI = Q(J,K) - (A(L,M)*BI + Q(L,M)*B)
      TEM = TD**2 + TDI**2
      TEMP = A(J,K)**2 + Q(J,K)**2
      TEMP = TEM / TEMP
      IF (II .EQ. 2) GO TO 92
      IF (II .EQ. 4) GO TO 92
      Q(J,K) = Q(J,K) * 10.D-18
      A(J,K) = A(J,K) * 10.D-18
      IF (TEMP .GT. 10.D-35) GO TO 92
      CNTR = CNTR + 1.
```

```
57            GO TO 90
58        92  A(J,K) = TD
59            Q(J,K) = TDI
60        90  GO TO (700,40,60,70),II
61         700 C    = DET*A(J,K) - DETI*Q(J,K)
62            DETI = DET*Q(J,K) + DETI*A(J,K)
63            DET = C
64            GO TO (30,100), II
65        100 CONTINUE
66            RETURN
67            END
```

```
1            SUBROUTINE RCOEF (K3)
2            IMPLICIT DOUBLE PRECISION (A-H,O-Z)
3            COMMON/INPUT/ Z(12), N, OMEGA,    V, VI, GCU(12), GSQ(12),
4           1 CAY(12), LAMBDA, LAMBDI, G(12)
5           2,RHO(12), GI(12), G SQI(12), CAYI(12)
6            DIMENSION RR(12), RI(12), RA(12), RT(12), CYSQ(12), CYSQI(12)
7            COMMON/REFL/ AF(12,200), AG(12,200), BF(12,200), BG(12,200),
8           2 EIGEN(350), EIGENI(350),BR(25,4), BI(25,4), CB(12), CBI(12),
9           3 CAYSQ(12), CAYSQI(12), NN
10           NM = N - 1
11           I = K3
12           IF (I .GT. NM) I = NM
13     110   J = I + I
14           K = J + 1
15           IF (NN .NE. 1) GO TO 102
16           L = I + 1
17           TEMP = CB(I)**2 + CBI(I)**2
18           CY = OMEGA * CB(I) / TEMP
19           CYI = -OMEGA * CBI(I) / TEMP
20           CYSQ(I) = CY**2 - CYI**2
21           CYSQI(I) = 2 * CY * CYI
22           CYSQI(I) = CYSQI(I) + CYSQI(I)
23     102   EL SQ = C YSQ(I) - EIGEN(NN)**2 + EIGENI(NN)**2
24           ELSQI = C YSQI(I) - 2.D0 * EIGEN(NN) * EIGENI(NN)
25           TEMP = ELSQ + DSQRT (ELSQ**2 + ELSQI**2)
26           IF (TEMP .LE. 0.D0) GO TO 107
27           EL = DSQRT (TEMP * .5D0)
28           ELI = ELSQI / (EL + EL)
29     103   A = AF(I,NN)*BR(J,2) - AG(I,NN)*BI(J,2)
30          * + BF(I,NN)*BR(J,1) - BG(I,NN)*BI(J,1)
31           B = AF(I,NN)*BI(J,2) + AG(I,NN)*BR(J,2)
32          * + BF(I,NN)*BI(J,1) + BG(I,NN)*BR(J,1)
33           E = AF(I,NN)*BR(K,2) - AG(I,NN)*BI(K,2)
34          * + BF(I,NN)*BR(K,1) - BG(I,NN)*BI(K,1)
35           F = AF(I,NN)*BI(K,2) + AG(I,NN)*BR(K,2)
36          * + BF(I,NN)*BI(K,1) + BG(I,NN)*BR(K,1)
37           C = (F * EL - E * ELI) / (ELSQ + ELSQI)
38           D = -(E * EL + F * ELI) / (ELSQ + ELSQI)
39           TEMP =  (A + C)**2 + (B + D)**2
40           RR(I) = (A**2 - C**2 + B**2 - D**2) / TEMP
41           RI(I) = -2.D0 * (A * D - B * C) / TEMP
42     10    FORMAT (10D13.5)
43           RA(I) = 0
44           IF (CB(I) .GT. V) GO TO 104
45           RX = CB(I) / V
46           RA(I) = ACOS(RX) * 57.296
47     104   RT(I) = RR(I)**2 + RI(I)**2
48           RT(I) = 1.D0 / RT(I)
49           RI(I) = -DATAN2 (RI(I), RR(I)) * 57.296D0
50           RR(I) = -4.34294D0 * DLOG (RT(I))
51           IF (K3 .NE.1) GO TO 108
52           I = I + 1
53           IF (I .LT. N) GO TO 110
54     105   CONTINUE
55           PRINT 106, (RR(I), I = 1,NM)
56           PRINT 106, (RI(I), I = 1,NM)
```

```
57            PRINT 106, (RT(I), I = 1,NM)
58            PRINT 106, (RA(I), I = 1,NM)
59       106  FORMAT (9G13.4)
60            RETURN
61       108  PRINT 109, Z(L), RR(I), RT(I), RI(I), RA(I)
62       109  FORMAT (9H AT DEPTH, F7.0,8H YD, R = ,F9.4 ,7H D8, OR,D12.4,
63           * 8H, PH A = ,F9.3,16H DEGREES, GR A =,F8.2,9H DEGREES. )
64            RETURN
65       107  EL = 0.D0
66            ELI = DSQRT ( DABS (ELSQ))
67            GO TO 103
68            END
```

```
      SUBROUTINE HANKEL (ZR, ZI, IH)
      COMMON /HAN/ H2R,H2I,H1R,H1I,H2PR,H2PI,H1PR,H1PI,R
      INTEGER FLPS, FLQUAD
      DOUBLE PRECISION ZMLA2, TLIM, R
      COMMON /LIMIT/ TLIM,EXPONT
      DOUBLE PRECISION   ZR,ZI,H2R,H2I,H1R,H1I,H2PR,H2PI,H1PR,H1PI,A0,
     1  A,B0,B,C0,C,D0,D,    C4,    D4,K,K1,K2,CON4,STORE1,STORE2,
     2  STORE3,STORE4,STORE5,STORE6,STORE7,STORE8,STORE9,STOR10,STOR11,
     3  STOR12,STOR13,STOR14,STOR15,STOR16,STOR18,C1,C2,C3,CPR,CPI,CTHR,
     4  CTP,FR,FI,FPR,FPI,          F1R,F1I,F2R,F2I,GR,GI,GPR,GPI,G1R,
     5  G1I,G2R,G2I,H11R,H11I,H12R,H12I,H11PR,H11PI,H12PR,H12PI,
     6  H21PR,H21PI,H22PR,H22PI,H21R,H21I,H22R,H22I,    PI,SR,SI,SPR,
     7  SPI,S2,   THR,X,XR,XI,XPR,XPI,YR,YI,         ZM,ZMSQ,ZM1R,ZM1I,
     8  EXPONT,A8K, ZRTM2R,ZRTM2I,ZRTM4R,ZRTM4I,ZRT2R,ZRT2I,ZRT2M,ZRT4R,
     9  ZRT4I,      Z32R,Z32I,STP,STOR17,STHR,C5,D5,FPI12
      DIMENSION A(40),B(40),C(40),D(40),C4(20),D4(20),ZMLA2(40),
     1 XPR(40), XPI(40), C5(20), D5(20), ZMLA5(20)
      DATA (A(I),I=1,36) /
     *           -1.5507278615487157D-001,    5.1690928718290523 7D-03,
     1   -7.1792956553181283 0D-05,    5.4388603449379756 0D-07,
     2   -2.5899334975895123 7D-09,    8.4638349594428508 9D-12,
     3   -2.0151987998673454 5D-14,    3.6507224635277997 3D-17,
     4   -5.2004593497547004 7D-20,    5.9775394824766672 0D-23,
     5   -5.6605487523453287 9D-26,    4.4924990097978799 0D-29,
     6   -3.0313758500660458 8D-32,    1.7603808653112926 1D-35,
     7   -8.8908124510671344 1D-39,    3.9409629658985524 9D-42,
     8   -1.5454756729013931 3D-45,    5.3999848808574183 5D-49,
     9   -1.6917245867347801 8D-52,    4.7788830133750852 7D-56,
     A   -1.2234723536546557 3D-59,    2.8519169082859107 8D-63,
     B   -6.0782542802342514 6D-67,    1.1890168779800961 4D-70,
     C   -2.1423727531172903 4D-74,    3.5670542009944894 1D-78,
     D   -5.5047132731396441 5D-82,    7.8954579362301264 3D-86,
     E   -1.0552603496698912 6D-89,    1.3174286512732724 9D-93,
     F   -1.5397716821800753 7D-97,    1.6883461427413107 2D-101,
     G   -1.7402042287583083 0D-105,   1.6891906705089383 6D-109,
     H   -1.5468779033964637 0D-113,   1.3385928551371267 8D-117/

      DATA (B(I),I=1,36) /
     *              -5.6524893762022989D-002,    1.3458308038576902 2D-03,
     1   -1.4953675598418780 2D-05,    9.5856894861658847 6D-08,
     2   -3.9940372859024519 9D-10,    1.1678471596206000 0D-12,
     3   -2.5278077048064935 0D-15,    4.2130128413441558 3D-18,
     4   -5.5727683086562907 8D-21,    5.9922239878024632 0D-24,
     5   -5.3406630907330331 6D-27,    4.0095068248746495 2D-30,
     6   -2.5701966826119548 2D-33,    1.4231432351118243 7D-36,
     7   -6.8750880923276539 8D-40,    2.9230816719080161 5D-43,
     8   -1.1022178250030226 8D-46,    3.7111711279563053 2D-50,
     9   -1.1225563000472792 9D-53,    3.0670937159761729 1D-57,
     A   -7.6068792558932860 0D-61,    1.7202350194240809 6D-64,
     B   -3.5615631872134181 3D-68,    6.7761856682142658 5D-72,
     C   -1.1888045031954852 4D-75,    1.9292510600381130 1D-79,
     D   -2.9046236977388030 9D-83,    4.0681004170011247 7D-87,
     E   -5.3136107850066938 0D-91,    6.4879252564184295 5D-95,
     F   -7.4215571452967622 4D-99,    7.9698852505334646 0D-103,
     G   -8.0503891419529945 5D-107,   7.6626586159841943 1D-111,
     H   -6.8846878345390325 0D-115,   5.8483594830563228 4D-119/
```

```
      DATA (C(I),I=1,36) /
     *          -3.101455723097431 4D-002,    6.461366089786315 47D-04,
     1   -6.526632413925571 18D-06,    3.884900246384268 56D-08,
     2   -1.523490292699713 16D-10,    4.231917479721425 44D-13,
     3   -8.761733912466719 35D-16,    1.404124024433769 13D-18,
     4   -1.793261844743000 16D-21,    1.867981088273958 50D-24,
     5   -1.617299543527236 80D-27,    1.182236581525757 89D-30,
     6   -7.393599634307428 97D-34,    4.000865602980210 48D-37,
     7   -1.891662223631305 19D-40,    7.881925931797104 97D-44,
     8   -2.915991835663005 91D-47,    9.642830144388247 05D-51,
     9   -2.867329808025051 16D-54,    7.707875828024331 08D-58,
     A   -1.882265159468701 12D-61,    4.193995453361633 51D-65,
     B   -8.560921521456692 20D-69,    1.606779564837967 75D-72,
     C   -2.782302276775701 74D-76,    4.458817751243111 76D-80,
     D   -6.632184666433306 21D-84,    9.180765042128053 99D-88,
     E   -1.185685786145945 24D-91,    1.431987664424747 10D-95,
     F   -1.620812297031658 29D-99,    1.722802186470725 22D-103,
     G   -1.722974483919117 13D-107,   1.624221798566286 89D-111,
     H   -1.445680283548096 92D-115,   1.216902595579206 16D-119/

      DATA (D(I),I=1,36) /
     *          -2.260995750480919 5D-001,    9.420815627003831 55D-03,
     1   -1.495367559841878 02D-04,    1.246139633201565 02D-06,
     2   -6.390459557443923 18D-09,    2.218909603279139 99D-11,
     3   -5.561178950574285 69D-14,    1.053253210336038 96D-16,
     4   -1.560375126423761 42D-19,    1.857589436218763 59D-22,
     5   -1.815825450849231 27D-25,    1.483517525203620 32D-28,
     6   -1.028078573044781 93D-31,    6.119515910980844 81D-35,
     7   -3.162540522470720 83D-38,    1.432310019234927 91D-41,
     8   -5.731532690015717 94D-45,    2.041144120375967 93D-48,
     9   -6.510826540274219 86D-52,    1.870927166745465 48D-55,
     A   -4.868402723771703 04D-59,    1.152557463014134 24D-62,
     B   -2.493094231043992 69D-66,    4.946615537796414 07D-70,
     C   -9.034914224285687 80D-74,    1.524108337430109 28D-77,
     D   -2.381791432145818 53D-81,    3.457885354450956 06D-85,
     E   -4.675977490805890 54D-89,    5.904011983340770 89D-93,
     F   -6.976263716578956 50D-97,    7.730788693017460 66D-101,
     G   -8.050389141952994 55D-105,   7.892538374463720 14D-109,
     H   -7.297770110461137 44D-113,   6.374711836531391 90D-117/

      DATA(C4(I), I=1,19) /
     *      .1041666666666666666D000,      -.5876374421296296294D000,
     *     -.2290716053934337712D001,      -.5115246914604383039D001,
     *     -.9062847663874030839D001,      -.1413420435039637896D002,
     *     -.2032957817611733257D002,      -.2764948541118776109D002,
     *     -.3609376712592949187D002,      -.4566262114618547916D002,
     *     -.5635611849738394099D002,      -.6817431262327333036D002,
     *     -.8111724483308463849D002,      -.9518494701182174927D002,
     *     -.1103774469890957246D003,      -.1266948822584689090D003,
     *     -.1441391353710093869D003,      -.1627327073751970376D003,
     *     -.1826444261146441383D003/
      DATA(D4(I), I=1,19) /
     *     -.1458333333333333333D000,      -.5242693865740740734D000,
     *     -.2190010740010626122D001,      -.4986228080782250417D001,
     *     -.8910269876251375731D001,      -.1396107513192384956D002,
     *     -.2013810597032928847D002,      -.2744104880120119211D002,
```

```
114           *     -.3586970305443466737D002,     -.4542393171194671533D002,
115           *     -.5610363644805757000D002,     -.6790874395729851569D002,
116           *     -.8083919802992951438D002,     -.9489495581082038481D002,
117           *     -.1100759893558574910D003,     -.1263823305425793323D003,
118           *     -.1438158255226323270D003,     -.1624125356051020867D003,
119           *     -.1826430810500566861D003/
120            DATA(C5(I), I=1,19) /
121           *     -.8020833333333333322D000,     -.2285545023696682453D001,
122           *     -.3778635359389885240D001,     -.5274623160711059862D001,
123           *     -.6771926170404923857D001,     -.8269954941340274075D001,
124           *     -.9768433620097970692D001,     -.1126721374768006806D002,
125           *     -.1276620743800279721D002,     -.1426535892246475774D002,
126           *     -.1576463088778327385D002,     -.1726399730395179650D002,
127           *     -.1876343942052581175D002,     -.2026294344140785724D002,
128           *     -.2176249668603070326D002,     -.2326204842605378820D002,
129           *     -.2476104285001339985D002,     -.2625430055269059493D002,
130           *     -.2772097924164600240D002/
131            DATA(D5(I), I=1,19) /
132           *     -.6770833333333333322D000,     -.2202914798206278015D001,
133           *     -.3712590308961503947D001,     -.5218010289498877822D001,
134           *     -.6721592615807936173D001,     -.8224186533352991837D001,
135           *     -.9726176955678393129D001,     -.1122776696710578781D002,
136           *     -.1272907520499096141D002,     -.1423017624892113601D002,
137           *     -.1573111965670700007D002,     -.1723193982027220702D002,
138           *     -.1873266140557309258D002,     -.2023330232819351955D002,
139           *     -.2173387494048982910D002,     -.2323435992394561375D002,
140           *     -.2473404635295374440D002,     -.2622252877088035571D002,
141           *     -.2758686258346552664D002/
142            DATA (ZMLA5(I), I=1,17) / 1.E9,715., 207., 103., 47., 36.4, 27.,
143           * 22.6, 18.5, 16.6, 14.7, 14., 12.9, 12.2, 11.5, 10.8, 9.2 /
144            DATA LA2, LA5 /36,17/
145            DATA (ZMLA2(I),I=1,40) /
146           12.6944301D-12,4.7348244D-6,7.0803713D-4,9.6398932D-3,
147           24.9271494D-2,1.5267301D-1,3.5324772D-1,6.7835277D-1,
148           31.1475215D0,1.7731141D0,2.5617749D0,3.9957181D0,5.2159327D0,
149           46.6020702D0,8.1490972D0,9.8514112D0,1.2320405D1,1.4917923D1,
150           51.7163634D1,1.9540309D1,2.3030246D1,2.6446844D1,2.9292820D1,
151           63.3549744D1,3.7541246D1,4.0802980D1,4.5784933D1,5.0287133D1,
152           75.3917166D1,5.9582285D1,6.4537858D1,7.0701377D1,7.5978472D1,
153           88.0163399D1,8.6945766D1,9.2594255D1,9.983446D1,
154           91.057519302,1.1039828D2,1.1820169D2/
155            DATA   C1 / 0.57735 026918962576D0  /
156            DATA   C2 / 0.66666 666666666666D0  /
157            DATA   C3 / 0.86602 54037844386 4D0  /
158            DATA   PI / 3.14159265358979324D0  /
159            DATA FPI12 / 1.30899693899574718D0/
160            DATA   CON4 / .70710678118654752 44D0/
161            A0 = 9.304367169292294 27D-01
162            B0 = 6.782987251442758 71D-01
163            C0 = 4.652183584646147 14D-01
164            D0 = 6.782987251442758 71D-01
165            K =   0.8536672188389515 6D0
166            ZMSQ = ZR*ZR +ZI*ZI
167            RZR = ZR
168            TEMP = ZI
169            IF (TEMP .LT. 0.) TEMP = -TEMP
170            IF (RZR .LE. 0.) GO TO 51
```

```
171             IF (RZR .GT. 4.4) GO TO 120
172             TEM1 = 7. - .2632 * RZR**2
173             IF (TEMP .GT. TEM1) GO TO 120
174             GO TO 53
175         51  IF (RZR .LT. -9.) GO TO 120
176             TEM1 = 4.4 + .1375 * RZR
177             IF (TEMP .GT. TEM1) GO TO 120
178         53  FLPS = 1
179             STORE1 = ZR*ZR-ZI*ZI
180             STORE2 = 2.*ZR*ZI
181             XR = STORE1*ZR -STORE2*ZI
182             XI = STORE1*ZI +STORE2*ZR
183             DO 55 MLS=1,LA2
184             IF (ZMSQ - ZMLA2(MLS)) 62,62,55
185         55  CONTINUE
186         62  FR = A0
187             FI = 0.0
188             XPR(1) = XR
189             XPI(1) = XI
190             DO 65 M = 1,MLS
191             FR=FR+A(M)*XPR(M)
192             FI=FI+A(M)*XPI(M)
193             XPR(M+1)=XR*XPR(M)-XI*XPI(M)
194             XPI(M+1)=XI*XPR(M)+XR*XPI(M)
195         65  CONTINUE
196             GR=B0
197             GI=0.0
198             DO 72 M = 1,MLS
199             GR=GR+B(M)*XPR(M)
200             GI=GI+B(M)*XPI(M)
201         72  CONTINUE
202             X =ZR*GR-ZI*GI
203             GI=ZR*GI+ZI*GR
204             GR=X
205             SR=-C1*(GI-FI-FI)
206             SI=C1*(GR-FR-FR)
207             H2R=GR-SR
208             H2I=GI-SI
209             GO TO 317
210        120  FLPS = 0
211             ZM = DSQRT(ZMSQ)
212             ZRT2M = DSQRT(ZM)
213             IF (ZR .LT. 0.D0) GO TO 125
214             ZRT2R = DSQRT (0.5D0 * (ZR + ZM))
215             ZRT2I = ZI / (ZRT2R + ZRT2R)
216             Z32R = ZR*ZRT2R - ZI*ZRT2I
217             Z32I = ZR*ZRT2I + ZI*ZRT2R
218             GO TO 130
219        125  ZRT2I = DSQRT (0.5D0 * (ZM - ZR))
220             IF (ZI .LT. 0.D0) ZRT2I = -ZRT2I
221             ZRT2R = ZI / (ZRT2I + ZRT2I)
222             Z32R = ZR*ZRT2R - ZI*ZRT2I
223             Z32I = ZR*ZRT2I + ZI*ZRT2R
224             ZM1R = DABS(Z32I)
225             IF (ZM1R .LT. TLIM) GO TO 130
226             R = (TLIM / ZM1R)
227             Z32R = Z32R * R
```

```
228             R = DCBRT(R)
229             ZRT2R = ZRT2R * R
230             ZRT2I = ZRT2I * R
231             ZRT2M = ZRT2M * R
232             R = R * R
233             ZM = ZM * R
234             ZMSQ = ZM**2
235             ZR = ZR * R
236             ZI = ZI * R
237         130 ZRT4R = DSQRT (0.5D0 * (ZRT2R + ZRT2M))
238             ZRT4I = 0.5D0 * ZRT2I / ZRT4R
239             ZRTM4R = ZRT4R/ZRT2M
240             ZRTM4I = -ZRT4I/ZRT2M
241             IF (ZR .GT. 0.) GO TO 210
242             IF (ZM1R .LT. TLIM) GO TO 210
243             ABK = ABS(K2)
244             IF (Z32I .GT. 0.) GO TO 205
245             K1 = K * EXPONT
246             K2 = K / EXPONT
247             Z32I = -TLIM
248             GO TO 220
249         205 K2 = K * EXPONT
250             K1 = K / EXPONT
251             Z32I = TLIM
252             GO TO 220
253         210 K2 = C2 * Z32I
254             S2 = DEXP(K2)
255             K2 = K*S2
256             K1 = K/S2
257         220 THR = FPI12 - C2 * Z32R
258             STHR =DSIN(THR)
259             CTHR =DCOS(THR)
260             STP = -C3*CTHR +0.5*STHR
261             CTP = -C3*STHR -0.5*CTHR
262             TEMP = DABS (Z32R)
263             TEM1 = DABS (Z32I)
264             IF (TEMP .LT. TEM1) TEMP = TEM1
265         230 DO 235  ML = 1,LA5
266             IF (TEMP .GT. ZMLA5(ML)) GO TO 250
267         235 CONTINUE
268         250 CONTINUE
269             YR = Z32I
270             YI = -Z32R
271             CALL CFR (YR, YI, F2R, F2I, C4, C5, ML)
272             CPR = F2R
273             CPI = F2I
274             STORE3=K2*(ZRTM4R*F2R-ZRTM4I*F2I)
275             STORE4=K2*(ZRTM4I*F2R+ZRTM4R*F2I)
276             H22R =STORE3*CTHR-STORE4*STHR
277             H22I =STORE3*STHR+STORE4*CTHR
278             IF (ZR) 280,270,270
279         270 FLQUAD =0
280             GO TO 300
281         280 IF (ZI) 290,310,310
282         290 FLQUAD = 1
283         300 H2R = H22R
284             H2I = H22I
```

```
285           GO TO 317
286      310  FLQUAD = -1
287           YR = -Z32I
288           YI = Z32R
289           CALL CFR (YR, YI, F1R, F1I, C4, C5, ML)
290           CPR = F1R
291           CPI = F1I
292           STORE5=K1*(ZRTM4R*F1R-ZRTM4I*F1I)
293           STORE6=K1*(ZRTM4R*F1I+ZRTM4I*F1R)
294           H21R=STORE5*CTP-STORE6*STP
295           H21I=STORE5*STP+STORE6*CTP
296           H2R=H21R+H22R
297           H2I=H21I+H22I
298      317  IF (IH .EQ. 2)GO TO 80
299      60   IF (FLPS .NE. 1) GO TO 320
300      70   H1R = GR+SR
301           H1I = GI+SI
302           GO TO 362
303      320  IF (FLQUAD .LT. 0)GO TO 340
304      330  YR = -Z32I
305           YI = Z32R
306           CALL CFR (YR, YI, F1R, F1I, C4, C5, ML)
307      340  STORE7=K1*(ZRTM4R*F1R-ZRTM4I*F1I)
308           STORE8=K1*(ZRTM4I*F1R+ZRTM4R*F1I)
309           H11R=STORE7*CTHR+STORE8*STHR
310           H11I=STORE7*(-STHR)+STORE8*CTHR
311           IF (FLQUAD .LE. 0) GO TO 360
312      350  STORE9=K2*(ZRTM4R*F2R-ZRTM4I*F2I)
313           STOR10=K2*(ZRTM4I*F2R+ZRTM4R*F2I)
314           H12R = STORE9*CTP+STOR10*STP
315           H12I = STORE9*(-STP)+STOR10*CTP
316           H1R = H11R+H12R
317           H1I = H11I+H12I
318           GO TO 362
319      360  H1R = H11R
320           H1I = H11I
321      362  IF (IH .EQ. 1)GO TO 999
322      80   IF (FLPS .NE. 1) GO TO 380
323      90   FPR = C0
324           FPI = 0.0
325           DO 92 M = 1,MLS
326           FPR=FPR+C(M)*XPR(M)
327      92   FPI=FPI+C(M)*XPI(M)
328           X  =-(STORE1*FPR-STORE2*FPI)
329           FPI=-(STORE1*FPI+STORE2*FPR)
330           FPR = X
331           GPR = D0
332           GPI = 0.0
333           DO 94 M = 1,MLS
334           GPR=GPR+D(M)*XPR(M)
335      94   GPI=GPI+D(M)*XPI(M)
336           SPR=-C1*(GPI-FPI-FPI)
337           SPI=C1*(GPR-FPR-FPR)
338           H2PR=GPR-SPR
339           H2PI=GPI-SPI
340           GO TO 414
341      380  YR =  Z32I
```

```
342             YI = -Z32R
343             CALL CFR (YR, YI, G2R, G2I, D4, D5, ML)
344             STOR11 = K2*(ZRT4R*G2R-ZRT4I*G2I)
345             STOR12 = K2*(ZRT4R*G2I+ZRT4I*G2R)
346             H22PR=STOR11*STHR+STOR12*CTHR
347             H22PI=STOR11*(-CTHR) +STOR12*STHR
348       390   IF (FLQUAD .LT. 0) GO TO 410
349       400   H2PR = H22PR
350             H2PI = H22PI
351             GO TO 414
352       410   YR = -Z32I
353             YI = Z32R
354             CALL CFR (YR, YI, G1R, G1I, D4, D5, ML)
355             STOR13 = K1*(ZRT4R*G1R-ZRT4I*G1I)
356             STOR14 = K1*(ZRT4R*G1I+ZRT4I*G1R)
357             H21PR=STOR13*(-STP) -STOR14*CTP
358             H21PI=STOR14*(-STP) +STOR13*CTP
359             H2PR = H21PR+H22PR
360             H2PI = H21PI+H22PI
361       414   IF (IH .EQ. 2) GO TO 999
362       100   IF (FLPS .NE. 1) GO TO 420
363       110   H1PR = GPR+SPR
364             H1PI = GPI+SPI
365             GO TO 999
366       420   IF (FLQUAD .LT. 0) GO TO 440
367       430   YR = -Z32I
368             YI = Z32R
369             CALL CFR (YR, YI, G1R, G1I, D4, D5, ML)
370       440   STOR15 = K1*(ZRT4R*G1R -ZRT4I*G1I)
371             STOR16 = K1*(ZRT4R*G1I +ZRT4I*G1R)
372             H11PR = STOR15*STHR -STOR16*CTHR
373             H11PI = STOR15*CTHR +STOR16*STHR
374       450   IF (FLQUAD .GT. 0) GO TO 470
375       460   H1PR = H11PR
376             H1PI = H11PI
377             GO TO 999
378       470   STOR17 = K2*(ZRT4R*G2R -ZRT4I*G2I)
379             STOR18 = K2*(ZRT4R*G2I +ZRT4I*G2R)
380             H12PR = STOR17*(-STP) +STOR18*CTP
381             H12PI = STOR17*(-CTP) -STOR18*STP
382             H1PR = H12PR+H11PR
383             H1PI = H12PI+H11PI
384       999   CONTINUE
385             RETURN
386             END

@PRT,S  J.CFR
```

```
1             SUBROUTINE CFR(X, Y, SR, SI, A, B, M)
2             IMPLICIT DOUBLE PRECISION (A-H,O-Z)
3             DIMENSION A(1), B(1)
4             SR = 0.D0
5             SI = 0.D0
6             DO 10 J = 1,M
7             I = M - J + 1
8             TEMR = X + SR + B(I)
9             TEMI = Y + SI
10            TEMP = A(I) / (TEMR**2 + TEMI**2)
11            SR = TEMR * TEMP
12            SI = -TEMI * TEMP
13      10    CONTINUE
14            SR = SR + 1.D0
15            RETURN
16            END
```

```
1             SUBROUTINE HZERO(PARTR,PARTI)
2             IMPLICIT DOUBLE PRECISION(A-H,O-Z)
3             DOUBLE PRECISION PARTI, PARTR, K2, IMZ12
4             COMMON /AHZERO/ HZEROR,HZEROI
5             D2 = PARTR**2 + PARTI**2
6           8 K2 = .7978845608*EXP(PARTI)
7             D = SQRT(D2)
8             RLZ12 = (SQRT((PARTR + D)/2.))/D
9             IF(D - PARTR) 9,9,10
10          9 IMZ12 = 0
11            GO TO 11
12         10 IMZ12 = (SQRT((D - PARTR)/2.))/D
13         11 COST = COS(PARTR)
14            SINT =-SIN(PARTR)
15            HZEROR = K2*(RLZ12*COST - IMZ12*SINT)
16            HZEROI = K2*(IMZ12*COST + RLZ12*SINT)
17            RETURN
18            END
```

## APPENDIX B: SAMPLE RUN

This appendix gives a brief discussion of the input-output, then lists an input deck and shows the resulting output. The input deck is really three separate runs that are stacked to run consecutively. The input to a single run consists of several parts given in table B1. The table gives the number of cards and the location of the FORTRAN input statements in Program MAIN. The last three of these are open-ended. That is, more modes, receiver depths, or ranges are read in until a blank card specifies the end of the list. A blank range card sends the program back to the beginning. A negative range sends the program back to read a new source and new receivers after reading another key card. The program halts when a blank "n and freq" card is encountered.

Table B2 gives most of the functions of the key card by which integers are read into control keys 1–9. Some of these will require additional information, which is read in immediately following the key card.

The output of the program is usually printed through FORTRAN print statements. Cards are also punched when key 5 is 10 or key 8 is 2, 3, or 4. In the first case each card contains a complete eigenvalue that can be read into future runs.

When key 8 = 2, propagation losses for 25 consecutive ranges per card are punched for each receiver depth, with a maximum of 5 receiver depths. The losses can be read into a plot program with a format of (5X,25F3.1). Each loss must then be subtracted from 140. This format allows losses to tenths of a dB from 40.1 to 140.0 dB.

When key 8 is 3, losses for up to 26 receiver depths are punched on one card for each range. These cards can be used in a contour plotting program. They can be read with a format of (26F3.1) and must also be subtracted from 140.

Table B1. Input cards to the normal mode program.

| Input | Function | Number of Cards | Location in Program MAIN |
|---|---|---|---|
| Control keys | Selects options | 1 or more | 37-65 |
| n and freq | Determines number of layers and frequency; also halts program | 1 | 66 |
| Profile | Specifies depths, sound speeds, gradients, attenuations, and densities | 7 | 71-85 |
| Modes | Searches for or specifies modes | 1 or more | 224 |
| Source and receivers | Specifies a source depth and one or more receiver depths | 2 or more | 461-463 |
| Ranges | Specifies a sequence of ranges; also directs continuation | 1 or more | 616 |

Table B2. Functions of the control keys.

| Key | Setting | Effect | Function Affected |
|---|---|---|---|
| 1 | 0 | No output | Depth functions |
| | | Print | |
| | | Plot on printer | |
| 2 | 0 | Print losses | Propagation losses |
| | 1 | Contour on printer | |
| 3 | 0 | No output | Reflection coefficients |
| | 1 | Print all interfaces | |
| | k > 1 | Print interface k | |
| 4 | k > 0 | Change levels and symbols | Contour on printer |
| 5 | 0 | Sum only those given | Number of modes |
| | 1 | Add to existing sum | |
| | k + 10 | Punch modes on cards | |
| 6 | 0 | Long print | Steps in mode iteration |
| | 1 | Short print | |
| | 2 | Shortest print | |
| 7 | 0 | Phased addition | Mode sum |
| | 1 | Random-phase addition | |
| 8 | 1 | Change T-lim | |
| | 2 | Punch losses for up to 5 receivers | Loss plot input |
| | 3 | Punch losses for up to 26 receivers | Contour plot input |
| 9 | 0 | No effect | Number of modes |
| | k | Use only 1st k modes | |

The first profile in the input-output is a surface duct, 100 m deep. For the 500 Hz frequency, 3 modes are found by searching from a phase velocity of 1520.5–1523 m/s. Two additional modes are found by extrapolation. Forty receiver depths are then specified from 3 to 120 m, and propagation loss contours are drawn for a source at a depth of 20 m. The modes are added in random phase, and loss contours of 80, 90, and 100 dB are requested to be represented by the symbols 8, 9, and 0. A negative range then causes the program to read new control keys, source and receivers. The depth functions are then printed out as amplitudes and phase angles and propagation losses are computed.

The second profile consists of two negative gradients over two layers of sediments in shallow water. A velocity discontinuity exists at the top of each sediment layer. Negative numbers in the input for the attenuation at the bottom of the sediment layers serve as flags to request that the gradients at the top of the layers have no imaginary parts and that the attenuation at the bottom will be whatever results from this. The change in ImC from 37.9 to 23.7 in the deeper sediment layer indicates that the attenuation changed by about 60 percent through the layer.

A final layer of negative sound speed gradient must always be added. A gradient of -0.1 is chosen here for the top of this layer.

The first three modes are determined by reading in approximate values. The magnitudes of the depth functions are plotted on a log scale at 2-m depths from 30 to 80 m. Reflection coefficients are computed at interface 2, which is the water-sediment interface.

The final profile is a deep-water profile with a 40-m deep sediment layer. The attenuation increases from 2 dB/km to 2.5 dB/km through this layer. The first mode, the first bottom-reflected mode, and a higher bottom-reflected mode are found. Each step of the mode iteration is printed out. Reflection coefficients are again computed. The amplitudes and phase of the depth functions are printed out at 500-m depth and at each even 1000-m depth for a 100-m source depth.

On the last two profiles, a much larger set of modes is required to compute correct propagation losses.

The sample run given here required 6 seconds on a UNIVAC 1110, Exec 8 operating system. The cost of the run was $1.20.

```
NORMAL*MODE(0).INPUT
     1       INPUT DECK STARTS AT LINE 3,  ENDS AT LINE 65.
     2      123456789 123456789 123456789 123456789 123456789 123456789 123456789 123456789
     3              1         1         1   1   2
     4      100.          90.         80.         -1000.
     5      0
     6      098
     7       2  500.
     8                    100.
     9      1520.
    10      0
    11      .017          -.1
    12      0
    13      0
    14      0
    15      -1520.5                   1523.                          30
    16      -1.                                                       2
    17      0
    18      10.
    19      3.            120.        3.
    20      0
    21      4000.         100000.     4000.
    22      -1.
    23         1
    24      10.
    25      30.           120.        30.
    26      0
    27      5000.         100000.     5000.
    28      0
    29         2          2             1
    30       5  1500.
    31      0.            51.         73.         73.3        373.3
    32      1542.2        1536.8      1606.45     1684.
    33                    1523.42
    34                                1.5         1.5         -.1
    35                                .12         .73         .73
    36                                -1.         -1.
    37                                1.68        1.91        1.91
    38      1527.18       .16
    39      1530.64       .13
    40      1533.49       .11
    41      0
    42      60.
    43      30.           80.         2.
    44      0
    45      0
    46         1          6
    47       8  100.
    48                    55.         146.        402.        960.        2286.       4390.       4430.
    49      1544.9        1542.6      1517.9      1495.0      1483.2      1497.8      1541.7
    50                                                                    1533.4
    51                                                                                1.          -.1
    52                                                                                .02         .025
    53                                                                                .025
    54                                                                                1.54        2.5
    55      -1483.5                   1484.5                         10
    58      -1533.4       .1          1534.4                         10
```

```
-1600.      .2         1602.                          10
0
100.
500.
1000.       5000.      1000.
0
0
0
0
123456789 123456789 123456789 123456789 123456789 123456789 123456789 123456789
```

```
KEYS    0   1   0   1   0   1   1   2   0
 2 LAYERS,      500.0 HZ
       .00000      100.00000
   1520.00000         .00000
       .00000         .00000
       .01700        -.10000
       .00000         .00000
       .00000         .00000
       .00000         .00000
ATTEN =.24539-001DB/KM
       RE M           IM M            L/KM          RE C           IM C       RE C BOTTOM      IM C BOTTOM          GRADIENT
      4.57175         .00000         .00000    1520.00000         .00000     1521.70286         .00000         .01700         .00000
      8.24353         .00000         .00000    1521.70286         .00000          .00000        .00000        -.10000         .00000
   1520.500          .000         .16061+003       .10879+002       .65010+001
   1520.583          .000         .48580+002       .59690+001       .35987+001
   1520.667          .000         .12071+002       .29662+001       .18090+001
   1520.750          .000         .20274+001       .12104+001       .74991+000
   1520.833          .000         .88666-001       .25086+000       .16043+000
   1520.917          .000         .65065-001      -.21599+000      -.13569+000
   1521.000          .000         .21841+000      -.39129+000      -.25554+000

   .15209166667+004     .00000000000        -.2160+000    -.1357+000     .2184+000     .0000
   .15208699858+004     .36285586937-003    -.9521-014    -.6003-014     .5345-022     .0000
Z =   233    0   -68    0
Z =  -223    0     0    0
 .000 .000  .000 .000
N=     1  LAMBDA =      .2065655+01   -.4928331-06 D=     -.4937188+02     .1698935+00 INT RANGE =       0. L/K =  .00428
   1520.953          .000         .14432+000      -.31942+000      -.20563+000
   1521.037          .000         .24401+000      -.41150+000      -.27328+000
   1521.120          .000         .21725+000      -.38317+000      -.26539+000
   1521.203          .000         .13931+000      -.30074+000      -.22105+000
   1521.287          .000         .68182-001      -.20341+000      -.16373+000
   1521.370          .000         .24079-001      -.11266+000      -.10671+000
   1521.453          .000         .47194-002      -.38741-001      -.56732-001
   1521.537          .000         .49825-003       .14984-001      -.16545-001
   1521.620          .000         .25922-002       .49104-001       .13453-001

   .15215366524+004     .36285586937-003     .1498-001    -.1655-001     .2592-002     .0000
   .15215320352+004     .33379340870-001     .3642-015     .2081-015     .1058-020     .0000
Z =   411   89   -14   27
Z =   -45   89     0    0
 .000 .000  .000 .000
N=     2  LAMBDA =      .2064756+01   -.4529658-04 D=     -.4419460+02     .7013564+01 INT RANGE =    6991. L/K =  .39345
   1521.615          .033         .20449-002       .38891-001       .23074-001
   1521.699          .033         .50536-002       .60096-001       .37975-001
   1521.782          .033         .66370-002       .67011-001       .46331-001
   1521.865          .033         .64766-002       .63289-001       .49711-001
   1521.949          .033         .51912-002       .52365-001       .49488-001
   1522.032          .033         .35740-002       .37214-001       .46789-001
   1522.115          .033         .22152-002       .20244-001       .42489-001
   1522.199          .033         .13974-002       .32976-002       .37236-001
   1522.282          .033         .11427-002      -.12314-001       .31482-001
   1522.365          .033         .13140-002      -.25737-001       .25527-001

   .15222820352+004     .33379340870-001    -.1231-001     .3148-001     .1314-002     .0000
   .15222575167+004     .21101619027+000     .6334-016     .2739-017     .1810-020     .0000
Z =   605  564    45  173
```

```
Z =   148  564     0    0
 .000 .000  .000 .000
N=     3  LAMBDA =     .2063772+01   -.2860812-03 D=    -.2457285+02    .8311493+01 INT RANGE =   6385. L/K = 2.48490
    1522.341         .211        .23202-003       -.15210-001       -.82123-003
    1522.424         .211        .75319-003       -.27433-001       -.77326-003
    1522.508         .211        .13283-002       -.36444-001       -.39025-003
    1522.591         .211        .17893-002       -.42300-001       -.92158-004

   .1523046302+004    .53327340407+000   -.2374-001    .1860-001    .1789-002    .0000
   .1523304832 9+004    .50934840311+000    .8377-015   -.3968-015    .3366-024    .0000
Z =   885 1360   131  418
Z =   428 1360     0    0
 .000 .000  .000 .000
N=     4  LAMBDA =     .2062353+01   -.6895903-03 D=    -.1425623+02    .5114411+01 INT RANGE =   4428. L/K = 5.98978

   .1524673984 0+004    .92837597939+000   -.7307-002   -.2996-003    .3366-024    .0000
   .1524728393 4+004    .88008242577+000    .1040-014   -.1501-014    .1148-016    .0000
Z =  1265 2344   248  721
Z =   808 2344     0    0
 .000 .000  .000 .000
N=     5  LAMBDA =     .2060427+01   -.1189291-02 D=    -.9215171+01    .3160372+01 INT RANGE =   3263. L/K =10.33018

SOURCE AND RECEIVERS
    10.00      3.00      6.00      9.00     12.00     15.00     18.00     21.00
    24.00     27.00     30.00     33.00     36.00     39.00     42.00     45.00
    48.00     51.00     54.00     57.00     60.00     63.00     66.00     69.00
    72.00     75.00     78.00     81.00     84.00     87.00     90.00     93.00
    96.00     99.00    102.00    105.00    108.00    111.00    114.00    117.00
   120.00
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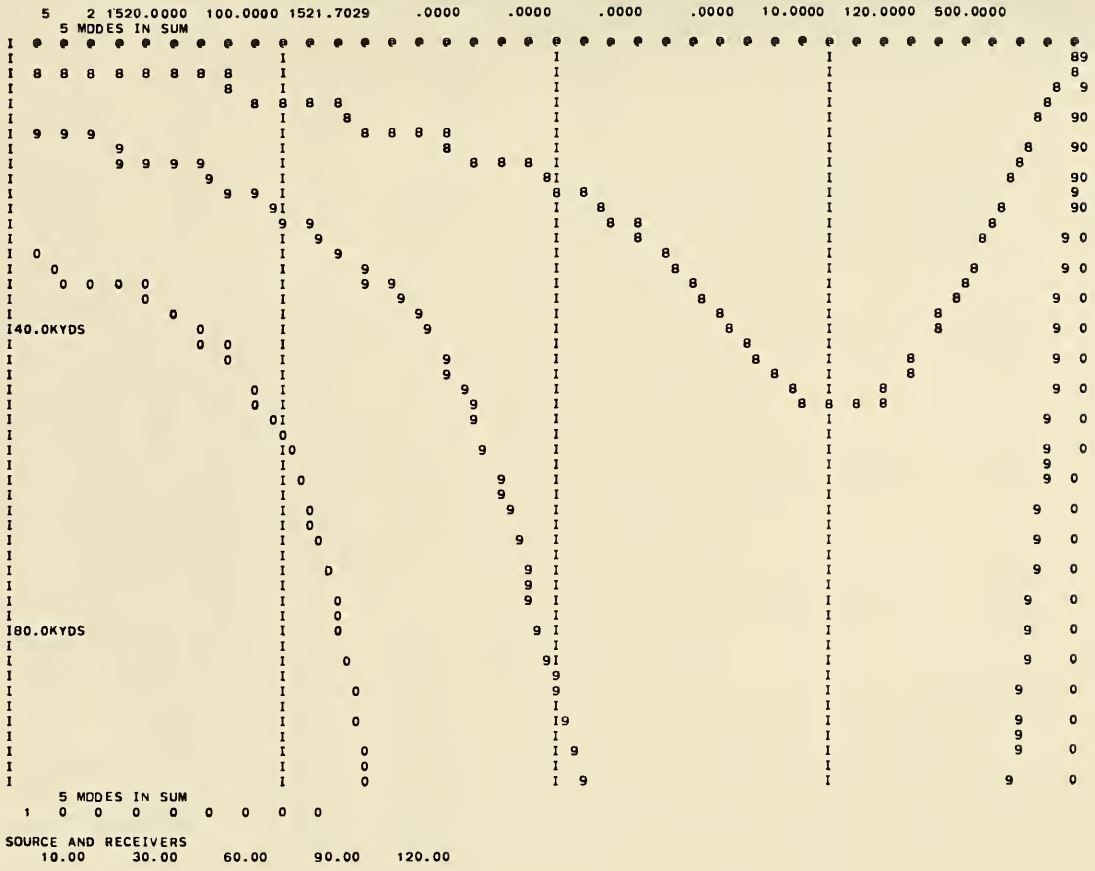
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5 MODES IN SUM
40.0KYDS
80.0KYDS
5 MODES IN SUM
1 0 0 0 0 0 0 0 0
SOURCE AND RECEIVERS
10.00 30.00 60.00 90.00 120.00

| | | | |
|---|---|---|---|
| .151-01 | .716-02 | .143-02 | .448-03 |
| .1725 | -.0318 | -4.0388 | -78.8365 |
| .625-02 | .128-01 | .847-02 | .403-02 |
| 5.3138 | 188.4864 | 172.8414 | 66.9433 |
| .244-02 | .768-02 | .197-01 | .146-01 |
| 269.9452 | 150.1511 | 2.1129 | 222.9555 |
| .138-01 | .221-01 | .316-01 | .358-01 |
| 209.0704 | 22.5897 | 201.6754 | 14.5748 |
| .213-01 | .221-01 | .455-01 | .677-01 |
| 186.2405 | -70.1573 | 42.6473 | 159.3411 |

5 MODES IN SUM

| | | | | |
|---|---|---|---|---|
| 5000. | 68.7 | 70.3 | 75.1 | 78.8 |
| 10000. | 74.3 | 72.1 | 77.7 | 84.2 |
| 15000. | 72.7 | 80.4 | 86.0 | 92.9 |
| 20000. | 74.2 | 85.2 | 89.7 | 94.5 |
| 25000. | 77.3 | 79.2 | 87.6 | 95.0 |
| 30000. | 77.5 | 81.7 | 91.7 | 100.8 |
| 35000. | 77.3 | 88.4 | 111.0 | 107.3 |
| 40000. | 78.8 | 84.8 | 95.4 | 102.0 |
| 45000. | 79.9 | 84.1 | 95.1 | 104.2 |
| 50000. | 79.8 | 87.1 | 102.6 | 119.6 |
| 55000. | 80.4 | 88.1 | 103.2 | 109.8 |
| 60000. | 81.3 | 86.8 | 98.8 | 107.5 |
| 65000. | 81.7 | 87.7 | 101.1 | 112.3 |
| 70000. | 81.9 | 89.2 | 105.1 | 115.9 |
| 75000. | 82.5 | 89.0 | 102.5 | 111.3 |
| 80000. | 83.0 | 89.0 | 102.2 | 112.2 |
| 85000. | 83.3 | 90.0 | 104.5 | 115.6 |
| 90000. | 83.7 | 90.4 | 104.8 | 114.4 |
| 95000. | 84.2 | 90.4 | 104.0 | 113.6 |
| 100000. | 84.5 | 90.9 | 104.8 | 115.2 |

5 MODES IN SUM

```
KEYS    2   0   2   0   0   1   0   0   0
 5 LAYERS,     1500.0 HZ
       .00000       51.00000       73.00000       73.30000      373.30000
   1542.20000     1536.80000     1606.00000     1684.00000         .C0000
       .00000     1523.42000        .00000         .00000         .C0000
       .00000        .00000       1.50000        1.50000        -.10000
       .00000        .00000        .12000         .73000         .73000
       .00000        .00000      -1.00000       -1.00000         .C0000
       .00000        .00000       1.68000        1.91000        1.91000
ATTEN =.99703-001DB/KM
       RE M           IM M            L/KM          RE C           IM C       RE C BOTTOM      IM C BOTTOM            GRADIENT
     -8.81029        .00000         .00000     1542.20000         .C0000     1536.80000         .00000       -.10644        .00000
     -6.84816        .00000         .00000     1536.80000         .C0000     1523.42000         .00000       -.61622        .00000
       .12021       -.00042      180.00229     1606.00000        5.E7131     1606.45019        5.67131       1.50000        .00000
    114.58114      -2.58229     1095.01391     1684.00000       37.E5189     2467.43825       23.72336       1.50000        .00000
     39.43319      -1.30290     1095.01391     2467.43825       81.E2581         .00000         .00000       -.10000        .00000

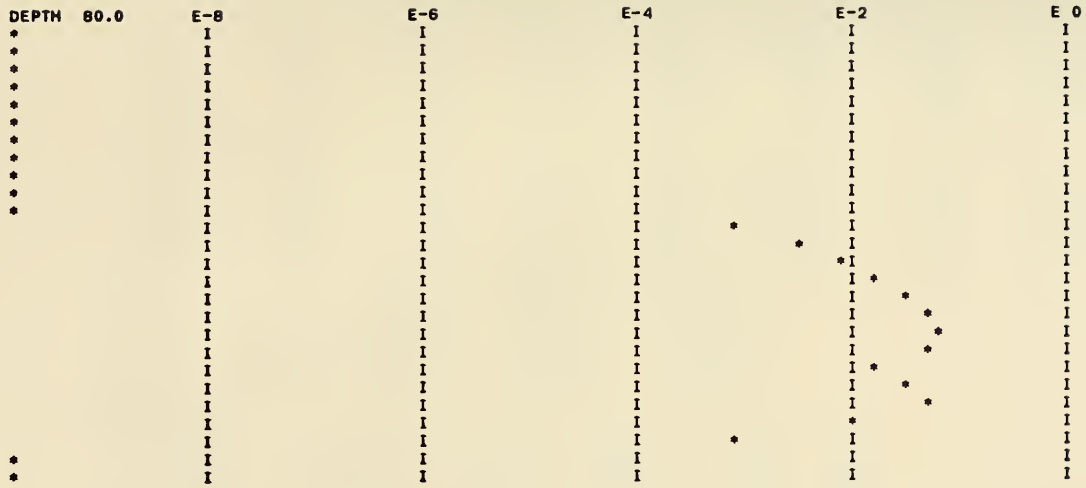
   .15271800000+004    .16000000000+000    -.2525-004    .2950-004    .1148-016    .0000
   .15271861943+004    .16198647471+000     .1348-008   -.6792-00E    .1197-013    .0000
Z =  -750  267  -490   83  -749-1469  -749-9189  -177*****
Z =  -750  267   194   83  -761-1469  -854-6607     0     0
 .237 .237  .237 .237  .237 .237  .237 .237  .237 .000
N=    1  LAMBDA =     .6171335+01    -.6545848-03 D=    -.2850630+C7    .1058118+07 INT RANGE =       0. L/K = 5.68572
AT DEPTH     73. YD, R =    1.9028 DB, OR  .6452+000, PH A =  113.538 DEGREES, GR A =    4.02 DEGREES.

   .15306400000+004    .13000000000+000     .1114-005   -.2117-00E    .1197-013    .0000
   .15306495580+004    .13096522218+000     .6485-010    .2047-00E    .2050-015    .0000
Z =  -750  215  -312   66  -749-1482  -749-9219  -176*****
Z =  -750  215   372   66  -761-1481  -854-6637     0     0
 .236 .236  .236 .236  .236 .236  .236 .236  .236 .000
N=    2  LAMBDA =     .6157371+01    -.5268402-03 D=    -.4104669+C4    .9252135+03 INT RANGE =     450. L/K = 4.57613
AT DEPTH     73. YD, R =    2.1761 DB, OR  .6059+000, PH A =   96.596 DEGREES, GR A =    5.57 DEGREES.

   .15334900000+004    .11000000000+000    -.2482-004    .1760-004    .2050-015    .0000
   .15334940269+004    .11306008086+000    -.4059-008    .3391-00E    .8069-012    .0000
Z =  -750  185  -167   57  -749-1490  -749-9242  -176*****
Z =  -543  185   517   57  -761-1489  -854-6660     0     0
 .235 .235  .235 .235  .235 .235  .235 .235  .235 .000
N=    3  LAMBDA =     .6145950+01    -.4531231-03 D=    -.2074773+C7    .6103755+06 INT RANGE =     550. L/K = 3.93583
AT DEPTH     73. YD, R =    2.2234 DB, OR  .5993+000, PH A =   87.310 DEGREES, GR A =    6.57 DEGREES.

SOURCE AND RECEIVERS
    60.00      30.00      32.00      34.00      36.00      38.00      40.00      42.00
    44.00      46.00      48.00      50.00      52.00      54.00      56.00      58.00
    60.00      62.00      64.00      66.00      68.00      70.00      72.00      74.00
    76.00      78.00      80.00
```

DEPTH 80.0
E-8
E-6
E-4
E-2
E 0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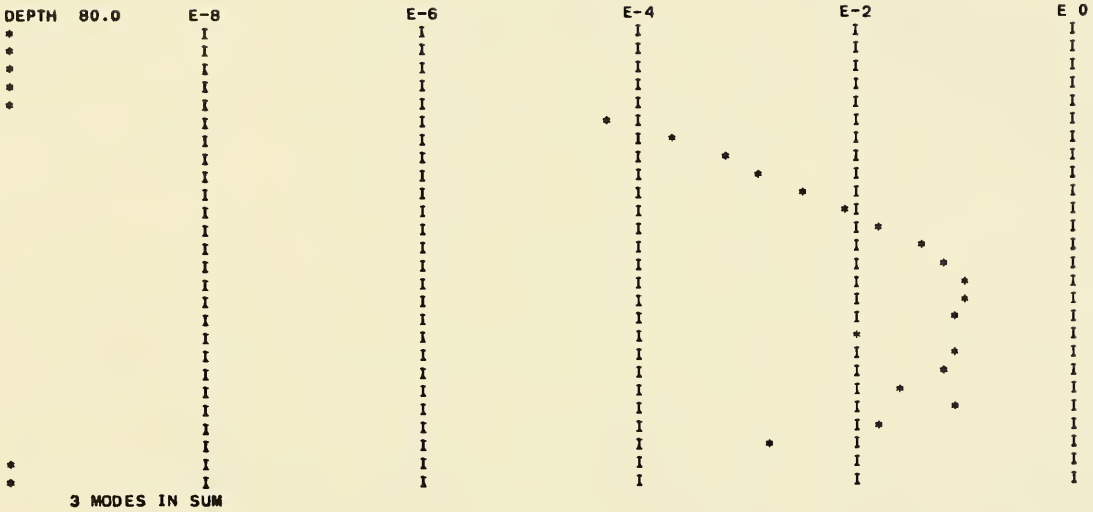
E-8
E-6
E-4
E-2
E 0


DEPTH 80.0
E-8
E-6
E-4
E-2
E 0
3 MODES IN SUM

```
KEYS    1   0   6   0   0   0   0   0   0
 8 LAYERS,      100.0 HZ
       .00000       55.00000      146.00000      402.00000      960.C0000     228E.00000     4390.00000     4430.00000
   1544.90000     1542.60000     1517.90000     1495.00000     1483.20000     1497.80000     1541.70000        .00000
       .00000         .00000         .00000         .00000         .C0000     1533.40000        .00000        .00000
       .00000         .00000         .00000         .00000         .C0000         .00000       1.00000       -.10000
       .00000         .00000         .00000         .00000         .C0000         .00000        .02000        .02500
       .00000         .00000         .00000         .00000         .C0000         .00000        .02500        .00000
       .00000         .00000         .00000         .00000         .C0000         .00000       1.54000       2.50000
ATTEN =.11895-002DB/KM
        RE M           IM M           L/KM           RE C           IN C        RE C BOTTOM     IM C BOTTOM           GRADIENT
     -1.14298         .00000         .00000     1544.90000         .C0000     1542.60000         .00000        -.04191        .00000
     -3.55888         .00000         .00000     1542.60000         .C0000     1517.90000         .00000        -.27809        .00000
     -7.02474         .00000         .00000     1517.90000         .C0000     1495.00000         .00000        -.09152        .00000
     -9.57769         .00000         .00000     1495.00000         .C0000     1483.20000         .00000        -.02140        .00000
     18.29300         .00000         .00000     1483.20000         .C0000     1497.80000         .00000         .01085        .00000
     32.94320         .00000         .00000     1497.80000         .C0000     1533.40000         .00000         .01634        .00000
      2.39805         .00378        2.00003     1541.70000         .E7104     1583.32718        1.14838        1.00000        .00642
    120.03173        -.08706        2.50003     1583.32718        1.14838         .00000         .00000        -.10000        .00000
   1483.500          .000        .31454+039      -.31203+019      -.17458+020
   1483.600          .000        .36434+039      -.33568+019      -.18790+020
   1483.700          .000        .38673+039      -.34568+019      -.19359+020
   1483.800          .000        .36481+039      -.33558+019      -.18803+020
   1483.900          .000        .28845+039      -.29827+019      -.16720+020
   1484.000          .000        .16645+039      -.22647+019      -.12701+020
   1484.100          .000        .42001+038      -.11371+019      -.63803+019
   1484.200          .000        .64709+037       .44612+018       .25044+019
   1484.300          .000        .20299+039       .24975+019       .14027+020

   .14842000000+004    .00000000000          .4461+018    .2504+019     .2030+039    .0000
   .14841653113+004    .15121756017-005   -.1565+018   -.8775+018     .2030+039    .0000
   .14841673537+004    .87643211278-006   -.1225+018   -.6873+016     .4171-005    .0000
   .14841747321+004    .23241154946-006    .1625+016    .9251+016     .5444-004    .0000
   .14841746342+004    .10000000000-017   -.1691+014   -.9489+014     .9596-008    .0000
   .14841746352+004    .24600381520-011   -.2477+010   -.1254+011     .9900-012    .0000
   .14841746352+004    .16732722238-014   -.1782+006   -.7745+005     .1743-019    .0000
Z =  -750    0  -750    0  -750    0  -750    0   123    0  -750    0  -365  -40  -749 -307
Z =  -750    0  -514    0  -343    0    79    0  -854    0  -854    0  -604  -44     0    0
 .235 .235  .235 .235  .235 .235  .235 .235  .235 .235  .235 .235  .235 .235  .235 .000
N=    1  LAMBDA =     .4233454+00    -.4772835-18 D=     -.1233262+28    -.1275428+23 INT RANGE =       0. L/K =  .00000
AT DEPTH  4390. YD, R =  -7.7133 DB, DR    .5907+001, PH A =     -.098 DEGREES, GR A =     1.00 DEGREES.
   1484.275          .000        .12168+039       .19339+019       .10860+020
   1533.400          .100        .42744-001      -.94064-001       .18411+000
   1533.500          .100        .60899-001       .28875-001       .24508+000
   1533.600          .100        .46658-001       .12032+000       .17995+000
   1533.700          .100        .21506-001       .14180+000       .37415-001
   1533.800          .100        .19682-001       .93327-001      -.10475+000
   1533.900          .100        .32457-001       .73283-002      -.18001+000

   .15338000000+004    .10000000000+000     .9333-001   -.1047+000     .3246-001    .0000
   .15337624954+004    .10000000000-017    -.3138-002   -.1881-001     .3246-001    .0000
   .15337554931+004    .10000000000-017    -.2196-002   -.1055-001     .1589-003    .0000
   .15337464534+004    .83226880921-003     .4164-004    .1464-003     .8241-004    .0000
   .15337465782+004    .81179352756-003    -.4671-006   -.9031-006     .1599-007    .0000
   .15337465774+004    .81209033462-003    -.6436-010   -.5979-010     .7043-012    .0000
   .15337466774+004    .81209038169-003    -.7988-014    .1180-013     .5258-020    .0000
Z =  -559    0  -125    0   467    0  2990    0  6112    0  3325    0   -48  -50  -750 -317
```

```
Z =  -444    0   230    0  1170    0  3948    0  4283    0    30    0  -287  -54     0    0
 .235 .235  .235 .235  .235 .235  .235 .235  .235 .235  .235 .235  .235 .235  .235 .000
N=     2  LAMBDA =     .4096625+00   -.2169087-06 D=     -.7141127+C6     .7150644+03 INT RANGE =     459. L/K =  .00188
AT DEPTH  4390. YD, R =     .0925 DB, DR   .9789+000, PH A =   153.731 DEGREES, GR A =     4.22 DEGREES.
    1533.847         .001         .10792-001        -.11971-001        -.10319+000
    1533.947         .001         .21413-001        -.16789-001        -.14537+000
    1534.047         .001         .13258-001        -.13092-001        -.11440+000
    1534.147         .001         .11330-002        -.36517-002        -.33462-001
    1600.000         .200         .60957-020         .77953-010         .43583-011
    1600.200         .200         .62615-020         .77626-010        -.15354-010
    1600.400         .200         .29078-020         .47172-010        -.26128-010
    1600.600         .200         .37426-021         .43201-012        -.19341-010
    1600.800         .200         .18268-020        -.42571-010         .38020-011

   .16006000000+004    .20000000000+000    .4320-012   -.1934-010    .1827-020    .0000
   .16006271212+004    .86853704778-001    .3449-011    .7868-011    .1827-020    .0000
   .16006062469+004    .92539871678-001    .7350-011    .5081-011    .4681-003    .0000
   .16006293949+004    .12555225080+000   -.1323-012   -.8078-013    .1626-002    .0000
   .16006289693+004    .12501210171+000    .3082-014   -.2074-015    .4729-006    .0000
   .16006289822+004    .12501608013+000    .8120-018   -.8123-01E    .1818-009    .0000
   .16006289822+004    .12501607809+000    .4661-023    .7166-023    .2514-016    .0000
Z =  2620   55   772   15  2291   31  7652   81 13327  126  8926   98   333  -56   461 -271
Z =  2734   55  1128   15  2993   31  8609   81 11498  126  5632   98    94  -59     0    0
 .235 .235  .235 .235  .235 .235  .235 .235  .235 .235  .235 .235  .235 .235  .235 .000
N=     3  LAMBDA =     .3925448+00   -.3065945-04 D=     -.1425299+C3     .5003842+00 INT RANGE =     367. L/K =  .26631
AT DEPTH  4390. YD, R =    6.3696 DB, DR   .2307+000, PH A =    -7.331 DEGREES, GR A =    16.66 DEGREES.
    1600.829         .125         .18299-020        -.38024-010         .19598-010
    1601.029         .125         .48884-020        -.58288-010         .38613-010
    1601.229         .125         .52029-020        -.55476-010         .46101-010
    1601.429         .125         .24051-020        -.34234-010         .35116-010
    1601.629         .125         .83588-022        -.60615-011         .68444-011

SOURCE AND RECEIVERS
   100.00     500.00    1000.00    2000.00    3000.00    4000.00    5000.00

    .172-20      .152-14      .000+00      .000+00      .000+00      .000+00
     -.0002       -.0002    269.9995     269.9995     269.9995     269.9995

    .345-03      .356-03      .249-03      .161-03      .650-03      .183-10
      .0574        .0053       -.1149    179.5611        .0729     28.8081

    .469-03      .431-03      .453-03      .376-03      .561-03      .242-03
     -.4893    182.1377     -3.6375    -13.4081       -.1417    175.5703
      3 MODES IN SUM



   KEYS    0   0   0   0   0   0   0   0   0

  @BRKPT PRINT$
```

## APPENDIX C: HANKEL FUNCTION PARAMETERS

This appendix gives the FORTRAN statements for two programs associated with the modified Hankel functions. Program PWRTRN computes the power series coefficients, $d_m$, from eq (57), then determines the truncation points from eq (59). The truncation points for the other three sets of coefficients can be determined by changing line 9. Different computer word lengths can be accommodated by changing line 16.

The second program, CFC, determines the asymptotic series coefficients $C_m$ from eq (72), then determines the continued fraction coefficients as indicated by eq (81)–(83). The second set of coefficients can be determined by changing the 4 in line 11 to a 16.

```
      PROGRAM PWRTRN
C ** THIS PROGRAM DETERMINS TRUNCATION POINTS FOR THE POWER SERIES.
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      DIMENSION D(50), ALOGD(50)
      D(1) = 1.
      ALOGD(1) = 0.
      P = 3.
      DO 50 I = 2,50
      D(I) = D(I-1) / P / (P-2)
      P = P + 3.
      IF (D(I) .LE 0.) GO TO 50
      ALOGD(I) = ALOG10(D(I))
   50 CONTINUE
      PRINT 60, D, ALOGD
   60 FORMAT (10E12.6)
      DH = 18.
      M = 1
      DO 10 K = 2,50
   30 P = M - K
      Z = (ALOGD(K) - ALOGD(M) + DH) / 3. / P
      IF (P .GT. -1.1) GO TO 20
      A = ALOGD(M) - ALOGD(M+1) - 3. * Z
      IF (A .GT. 0.) GO TO 20
      M = M + 1
      GO TO 30
   20 L = K - 1
      MM = M - 1
      AZ = EXP (Z * 2.3025851)
      AZSQ = AZ * AZ
      PRINT 40, L, MM, Z, AZ, AZSQ
   40 FORMAT (2I5, 4E15.8)
   10 CONTINUE
      END
```

```
      PROGRAM CFC
C ** THIS PROGRAM COMPUTES A SET OF SERIES COEFFICIENTS AND THEN
C ** COMPUTES THE CORRESPONDING CONTINUED FRACTION COEFFICIENTS.
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      DIMENSION COEF(21,23,3), CHECK(20), C(82), S(10), A(20), B(20)
      C(1) = 1.
      BOTTOM = 1.
      TOP = 1.
      DO 2 I = 1,45
      X = 48 * I
      Y = 9 * (I + I - 1)**2 - 4
      C(I+1) = C(I) * Y / X
    2 CONTINUE
      PRINT 20, (C(I), I = 1,40)
   20 FORMAT (5G20.9)
   11 FORMAT (/)
      DO 100 I = 1,11
      COEF(I,I,3) = 0.
      COEF(I,I+1,3) = 0.
      COEF(I,I+2,3) = 0.
  100 CONTINUE
      A(1) = C(2)
      COEF(2,2,3) = 1.
      DO 140 I = 3,21
      DO 110 J = 2,I
      COEF(I,J,1) = COEF(I-1,J,3)
      COEF(I,J,2) = COEF(I-2,J,3)
      COEF(I,J,3) = COEF(I-1,J-1,3)
  110 CONTINUE
      IF (I .EQ. 3) GO TO 150
      CON = 0.
      AT = 0.
      BT = 0.
      K = I - 3
      DO 120 J = 3,I
      K = K + 1
      CON = C(K) * COEF(I,J-1,3) + CON
      AT = C(K) * COEF(I,J-1,2) + AT
      BT = C(K) * COEF(I,J-1,1) + BT
  120 CONTINUE
      PRINT 160, CON, AT, BT
      CHECK(I-2) = BT
      A(I-2) = -(CON + C(K+1)) / AT
  150 CONTINUE
      CON = 0.
      AT = 0.
      BT = 0.
      K = I - 2
      DO 130 J = 3,I
      K = K + 1
      CON = C(K) * COEF(I,J-1,3) + CON
      AT = C(K) * COEF(I,J-1,2) + AT
      BT = C(K) * COEF(I,J-1,1) + BT
  130  CONTINUE
      PRINT 160, CON, AT, BT
      PRINT 11
```

```
57            B(I-2) = -(CON + A(I-2) * AT + C(K+1)) / BT
58            DO 140 J = 2,I
59            COEF(I,J,3) = COEF(I,J,3) + A(I-2) * COEF(I,J,2) + B(I-2) *
60           * COEF(I,J,1)
61       140  CONTINUE
62            PRINT 20, A, B, CHECK
63       160  FORMAT (5G20.9)
64            K = -2
65            J = 0
66            DO 30 M = 1,18,3
67            J = J + 3
68            K = K + 3
69            PUNCH 200, (A(I), I = K, J)
70            PUNCH 200, (B(I), I = K, J)
71       30   CONTINUE
72       200  FORMAT (5X, 1H*, 3(E21.15, 1H, ))
73            END
```

## APPENDIX D: MODE FOLLOWER PROGRAM IN FORTRAN

The FORTRAN statements of the Mode Follower program are given here. This is the main body of the program. The following auxiliary subroutines from appendix A are required: SETUP, DET, HANKEL, and CFR.

```
1             PROGRAM MFOLLO
2             IMPLICIT DOUBLE PRECISION (A-H,O-Z)
3             COMMON /INPUT/ Z(10),N,OMEGA,V,VI,GCU(10),GSQ(10),CAY(10),LAMBDA,L
4            1AMBDI,G(10),RHO(10),GI(10),GSQI(10),CAYI(10)
5             COMMON /DETMNT/ A(21,4),Q(21,4)
6             REAL INCA, INCB, INCC, INCD, INCE, LAMBDA, LAMBDI
7             DIMENSION T(4), PV(4), W(8), WI(8), CB(10), CBI(10), C(10),
8            1  CAY SQ(10), GAMMA(10), DPK(10), GCUI(10), CI(10), CR(10),PVI(4)
9            2 , CAYSQI(10), SR(4), SI(4)
10            CHNG = 1. / 8192.
11            CHNGI = 0.
12       4    CONTINUE
13      C** K0 - TOTAL STEP LIMIT, K1, K2 PRINT KEYS, K3 = 1 KEEPS SAME PROFILE
14      C**     FOR NEXT RUN.
15            READ 10, K0, K1, K2, K6, K3, TLIM, BLIM, RATIO, EX
16            PRINT 10,K0, K1, K2, K6, K3, TLIM, BLIM, RATIO, EX
17       10   FORMAT (5I4, 4E10.1)
18            IF (TLIM .EQ. 0.) TLIM = 1.E-5
19            IF (BLIM .EQ. 0.) BLIM = 1.E-2
20            IF (EX .EQ. 0.) EX = 28.
21            RLIM = 10.**EX
22            IF (RATIO .EQ. 0.) RATIO = 2.
23            IF (K0 .EQ. 0) K0 = 300
24            IF (K3 .NE. 0) GO TO 128
25      30    READ 1240, N,FREQ    ,ATTEN
26      C** STOP IF N = 0. THIS IS THE ONLY PROGRAMED STOP.
27            IF (N.EQ.0) GO TO 1200
28            PRINT 1250, N,FREQ
29      C** PARAMETERS READ IN BELOW ARE THOSE AT THE TOP OF EACH LAYER.
30      C** READ IN VELOCITIES.
31            READ 1260, (C(I),I=1,N)
32            PRINT 1280, (C(I),I=1,N)
33      C** READ IN DEPTHS.
34            READ 1260, (Z(I),I=1,N)
35            PRINT 1280, (Z(I),I=1,N)
36      C** READ IN GRADIENTS
37            READ 1260, (GAMMA(I),I=1,N)
38            PRINT 1280, (GAMMA(I),I=1,N)
39      C** READ IN ATTENUATION FACTOR IN LOSS PER KILOMETER.
40            READ 1260, (DPK(I),I=1,N)
41            PRINT 1280, (DPK(I),I=1,N)
42      C** READ IN DENSITIES (BLANK INPUT IMPLIES SEA WATER DENSITY).
43            READ 1260, (RHO(I),I=1,N)
44            PRINT 1280, (RHO(I),I=1,N)
45       128  CONTINUE
46            NUMBER = 1
47            JX = 0
48      C**   NX = VARIABLE, NY = LAYER NUMBER, NZ = CONTINUITY
49            READ 119, NX, NY, NZ, PK, VALL,DP, V, VI, STEP, STEPI
50            PRINT 21, NX, NY, NZ, PK, VALL,DP, V, VI, STEP, STEPI
51       119  FORMAT (3I2, 4X, 7D10.2)
52       21   FORMAT (10H  VARIABLE ,I2, 10H  LAYER NO , I2,12H  CONTINUITY
53           * I2, / 7G15.5)
54            PK = PK - OP
55      C** START NEW CYCLE BY INCREMENTING VARIABLE.
56       107  PK = PK + DP
```

```
      IF (DP) 108,999,109
C** CHECK IF DESIRED LIMIT OF VARIABLE HAS BEEN REACHED.
 108  IF (PK .LT. VALL) GO TO 3
      GO TO 133
 109  IF (PK .GT. VALL) GO TO 3
 133  GO TO (131,101,102,103,104,105),NX
 131  FREQ = PK
      GO TO 106
 101  C(NY) = PK
      IF (NZ .NE. 0) GO TO 106
 134  IF (NY .EQ. N) GO TO 135
      GAMMA(NY) = 0.
      IF (NY .LT. 2) GO TO 106
 135  GAMMA(NY-1) = 0.
      GO TO 106
 102  Z(NY) = PK
      IF (NZ .EQ. 1) GO TO 106
      IF (NY .LT. N) GO TO 134
      IF (NUMBER .EQ. 1) GO TO 106
      C(NY) = 0.
      GO TO 106
 103  GAMMA(NY) = PK
      IF (NZ .NE.0) GO TO 106
      J = NY + 1
      DO 121 I = J,N
      C(I) = 0.
 121  CONTINUE
 104  DPK(NY) = PK
      GO TO 106
 105  RHO(NY) = PK
 106  CONTINUE

C** COMPLETE PROFILE **
      DO 100 I=1,N
C** SET UNSPECIFIED DENSITIES TO 1.02 (SEA WATER).
        IF (RHO(I).NE.0.) GO TO 40
        RHO(I)=1.02
40      IF (I.EQ.1) GO TO 50
C**       COMPUTE VELOCITY AT BOTTOM OF PREVIOUS LAYER.
        TEMP=CI(I-1)**2
        TEMDR=C(I-1)**2
        TEMDI=(TEMDR+TEMDR+TEMDR-TEMP)*CI(I-1)
        TEMDR=(TEMDR-TEMP-TEMP-TEMP)*C(I-1)
        TEMP=(GAMMA(I-1)+GAMMA(I-1))*(Z(I)-Z(I-1))-C(I-1)
        TEMDEN=TEMP**2+CI(I-1)**2
        TEM1=(TEMDI*CI(I-1)-TEMDR*TEMP)/TEMDEN
        TEM1I=(-TEMDI*TEMP-TEMDR*CI(I-1))/TEMDEN
      CB(I)=SQRT(.5*(TEM1+SQRT(TEM1**2+TEM1I**2)))
        CBI(I)=TEM1I/(CB(I)+CB(I))
50      IF (C(I).NE.0) GO TO 60
C** IF VELOCITY WAS UNSPECIFIED USE VELOCITY AT BOTTOM OF PREVIOUS LAYER
        C(I)=CB(I)
60      IF (DPK(I).NE.0.) GO TO 70
        CI(I)=0.
        GO TO 80
C** IF ATTENUATION IS TO BE APPLIED TO A LAYER, COMPUTE COMPLEX VELOCITY
C** KEEP ABSOLUTE C EQUAL TO GIVEN REAL C FOR SIMPLICITY.
```

```
      70    TEMP=54578.*FREQ
              TEMDI=DPK(I)*C(I)
              TEMDR=TEMP**2+TEMDI**2
              CI(I)=TEMDI*TEMP*C(I)/TEMDR
              C(I)=TEMP**2*C(I)/TEMDR
      80      IF (GAMMA(I).NE.0.) GO TO 100
              IF (I.EQ.N) GO TO 90
      C** COMPUTE GRADIENT IF NOT GIVEN.
              GAMMA(I)=(C(I+1)**2-C(I)**2)*C(I)/(2.*C(I+1)**2*(Z(I+1)-Z(I)))
              IF (I.EQ.N) GO TO 90
              GO TO 100
      C** REDUCE LAYERS BY ONE IF FINAL POINT ONLY DEFINES GRADIENT IN LAST LAYER.
      90      N=N-1
      100     CONTINUE

      C** COMPUTE USEFULL QUANTIES **
            OMEGA=6.283185307*FREQ
            DO 120 I=1,N
              TEMP=C(I)**2+CI(I)**2
              CAY(I)=OMEGA*C(I)/TEMP
              CAYI(I)=-OMEGA*CI(I)/TEMP
              CAYSQ(I)=CAY(I)**2-CAYI(I)**2
            CAYSQI(I)=2.*CAY(I)*CAYI(I)
              TEMDR=-2.*GAMMA(I)*CAYSQ(I)
            TEMDI=-2.*GAMMA(I)*CAYSQI(I)
              GCU(I)=(TEMDR*C(I)+TEMDI*CI(I))/TEMP
              GCUI(I)=(TEMDI*C(I)-TEMDR*CI(I))/TEMP
            TEMP=EXP(ALOG(GCU(I)**2+GCUI(I)**2)/6.)
            GI(I)=TEMP*SIN(ATAN2(GCUI(I),ABS(GCU(I)))/3.)
            G(I)=SQRT(TEMP**2-GI(I)**2)
              IF (GAMMA(I).LT.0.) GO TO 110
              G(I)=-G(I)
      110     GI(I)=-GI(I)
      C** XM IS A LAYER STRENGH PARAMETER USED ONLY TO COMPARE WITH OTHER MODE
              XMI=-GI(I)*(Z(I+1)-Z(I))
              XM=-G(I)*(Z(I+1)-Z(I))
              GSQI(I)=2.*G(I)*GI(I)
      120     GSQ(I)=G(I)**2-GI(I)**2
            IF (JX .GT. 0) GO TO 113
      C** GO TO INITIAL 3 STEPS OR TO THE STANDARD STEP.
            IF (NUMBER - 4) 71,111,122
       71   CALL SETUP
            CALL DETNT(N,DET,DETI)
              VEL=DET
              VELI=DETI
              DELTA=STEP
              DELTI=STEPI
              IF(DELTA.NE.0.)GO TO 250
              IF(DELTI.EQ.0.)DELTA=.01
      250     SIZE2=100.
              IF (K6.LT.3) PRINT 1320, V,VI,DET,DETI,A(21,4),Q(21,4)
      C** ITERATE FOR MODE UP TO 7 STEPS.
              DO 310 J=1,12
                V=V+DELTA
                VI=VI+DELTI
      C** DO NOT PERMIT IMAGINARY PART TO BECOME NEGATIVE.
                IF (VI) 260,270,280
```

```
      260       DELTI=DELTI-VI
      270   VI=1.E-18
      C** SET UP DETERMINANT FOR PHASE VELOCITY V + VI
      280       CALL SETUP
                CALL DETNT (N,DET,DETI)
                IF (K6.NE.1) GO TO 300
            PRINT 1330, V, VI, DET, DETI, SLR, SLI
       300  TEMNR = DELTA
            TEMNI = DELTI
                TEMDR=VEL-DET
                TEMDI=VELI-DETI
                 TEMDEN=TEMDR*TEMDR+TEMDI*TEMDI
                IF (TEMDEN.EQ.0.) GO TO 320
                TEMRNU=TEMNR*TEMDR+TEMNI*TEMDI
                TEMINU=TEMNI*TEMDR-TEMNR*TEMDI
            SLR = TEMRNU / TEMDEN
            SLI = TEMINU / TEMDEN
            IF (J .GT. 3) GO TO 125
     *      SR(4-NUMBER) = SLR
            SI(4-NUMBER) = SLI
       125  DELTA = DET * SLR - DETI * SLI
            DELTI = DET * SLI + DETI * SLR
                SIZE=DELTA*DELTA+DELTI*DELTI
      C** DISCONTINUE ITERATION AFTER 2ND STEP IF CORRECTION STEP IS MORE THAN
      C** PREVIOUS STEP.
                IF ((SIZE.GT.SIZE2).AND.(J.GT.2)) GO TO 320
                SIZE2=SIZE*2.
                VEL=DET
                VELI=DETI
      310       CONTINUE
       320    CONTINUE
       51    PV(4-NUMBER) = V
            PVI(4-NUMBER)= VI
            NUMBER = NUMBER + 1
            GO TO 107
      C** START STANDARD STEP, EXTRAPOLATE PHASE VELOCITY AND SLOPE.
       111  INCA = DP
            INCB = DP
            INCC = DP
       122  INCD = -INCB - INCC
            T(1) = INCB * INCD
            T(2) = INCB * INCC
            T(3) = INCD * INCC
            DO 112 IS = 1,3
            W(IS+4) = -SR(IS) / T(IS)
            WI(IS + 4) = -SI(IS) / T(IS)
            W(IS) = -PV(IS) / T(IS)
       112  WI(IS) = -PVI(IS) / T(IS)
       113  INCD = INCA + INCB
            INCE  = INCD + INCC
            T(1) = INCD * INCE
            T(2) = INCA * INCE
            T(3) = INCA * INCD
            SLOP = 0.
            SLOPI = 0.
            SUM = 0.
            SUMI = 0.
```

```
228           DO 114 IS = 1,3
229           SLOP = SLOP + W(IS + 4) * T(IS)
230           SLOPI = SLOPI + WI(IS+4) * T(IS)
231           SUM = SUM + W(IS) * T(IS)
232       114 SUMI = SUMI + WI(IS) * T(IS)
233           V = SUM
234           VI = SUMI
235           CALL SETUP
236           CALL DETNT (N,DET,DETI)
237     C** EVALUATE DETERMINANT AT THE EXTRAPOLATED POINT.
238           VEL = DET
239           VELI = DETI
240     C** ITERATE FOR THE ROOT USING EXTRAPOLATED SLOPE.
241           DELTA = DET * SLOP - DETI * SLOPI
242           DELTI = DET * SLOPI + DETI * SLOP
243           IF (K1 .EQ. 1) PRINT 1330, V, VI, DET, DETI, DELTA, DELTI
244           V = V + DELTA
245           VI = VI + DELTI
246           IF (VI .GE. 0.) GO TO 124
247           DELTI = DELTI - VI
248           CHNGI = CHNGI - VI
249           VI = 0.
250     C** RE-EVALUATE AT NEW POINT.
251       124 CALL SETUP
252           CALL DETNT (N, DET, DETI)
253           TEMNR = DELTA
254           TEMNI = DELTI
255               TEMDR=VEL-DET
256               TEMDI=VELI-DETI
257                TEMDEN=TEMDR*TEMDR+TEMDI*TEMDI
258           IF (TEMDEN .EQ. 0.) GO TO 123
259               TEMRNU=TEMNR*TEMDR+TEMNI*TEMDI
260               TEMINU=TEMNI*TEMDR-TEMNR*TEMDI
261     C** EVALUATE SLOPE (RECIPROCAL ACTUALLY USED).
262           SLR = TEMRNU / TEMDEN
263           SLI = TEMINU / TEMDEN
264           DELTA = DET * SLR - DETI * SLI
265           DELTI = DET * SLI + DETI * SLR
266           IF (K1 .EQ. 1) PRINT 1330, V, VI, DET, DETI, DELTA, DELTI
267     C** CORRECT PHASE VELOCITY TO BEST VALUE.
268           V = V + DELTA
269           VI = VI + DELTI
270           TEMP = V**2 / (TEMNR**2 + TEMNI**2)
271     C** WAS INCREMENT LARGE ENOUGH TO PERMIT EVALUATION OF SLOPE.
272           IF (TEMP .LT. RLIM) GO TO 123
273           IF (TEMP .LT. 1.E34) GO TO 141
274           SLR = SLOP
275     C** IF NOT, USE EXTRAPOLATED SLOPE.
276           SLI = SLOPI
277           GO TO 141
278       123 CONTINUE
279     C** IF SO, FIND 1 - RATIO OF SLOPES.
280           TEMDEN = (SLR**2 + SLI**2)
281           TEMDR = SLR * SLOP + SLI * SLOPI - TEMDEN
282           TEMDI = SLR * SLOPI - SLI * SLOP
283           TEMP = (TEMDR**2 + TEMDI**2) / TEMDEN**2
284           IF (TEMP .GT. TLIM) GO TO 116
```

```
C** SLOPE RATIO TOO GOOD.  DOUBLE STEP.
 141  DP = DP * RATIO
      GO TO 117
 116  IF (TEMP .LT. 8LIM) GO TO 117
      PRINT 130, PK,V,VI,DET,DETI,SLR,SLI,TEMP,DBLOSS,NUMBER
 130  FORMAT (1X,E14.6,E16.9,E13.7,6E10.3,I5)
C** SLOPE RATIO TOO POOR.  HALVE STEP.
      IF (NUMBER .LT. 7) GO TO 126
      PK = PK - DP
      DP = DP / RATIO
      INCA = DP
      JX = JX + 1
      IF (K2 .EQ. 1) PRINT 118, PK, V, VI, DET, DETI
C** STOP ON 5 SUCCESSIVE FAILURES.  MODE IS LOST.
      IF (JX .LT. 5) GO TO 107
      PRINT 810, N, FREQ
 810  FORMAT (I4, G12.5)
 3    DO 801 I = 1,N
      PRINT 800, C(I), Z(I), GAMMA(I), DPK(I), RHO(I), G(I)
 800  FORMAT (10G12.5)
 801  CONTINUE
      GO TO 4
 126  PRINT 127 , N, TEMP
 127  FORMAT (7H NUMBER, I3, 22H FAILED, SLOPE RATIO  ,F10.6)
C** UPDATE ALL QUANTITIES FOR NEXT STEP.
 117  INCC = INCB
      INCB = INCA
      INCA = DP
      PV(3) = PV(2)
      PVI(3) = PVI(2)
      PV(2) = PV(1)
      PVI(2) = PVI(1)
      PV(1) = V
      PVI(1) = VI
      JX = 0
      DENOM = V * V + VI * VI
      LAMBDI = -OMEGA * VI / DENOM
      DB LOSS = -8686. * LAMBDI
      SR(3) = SR(2)
      SR(2) = SR(1)
      SR(1) = SLR
      SI(3) = SI(2)
      SI(2) = SI(1)
      SI(1) = SLI
      GV = V**2 / (V - FREQ * (V-PV(2)) / INCB)
      PRINT 118, PK,V,VI,DET,DETI,SLR,SLI,TEMP,DBLOSS,GV,NUMBER
 118  FORMAT (E15.7,E16.9,E13.7,6E10.3,F11.5,I5)
      NUMBER = NUMBER + 1
C** CHECK TOTAL NUMBER OF STEPS.
      IF (NUMBER .GT. K0) GO TO 3
      GO TO 107
 999  STOP
1250  FORMAT (I3,8H LAYERS,,F10.1,3H HZ)
1260  FORMAT(6E10.4)
1270  FORMAT (8H ATTEN =,G10.5,5HDB/KM)
1280  FORMAT (8F14.5)
1320      FORMAT (/,6E18.9)
```

```
342      1330  FORMAT (6E18.9)
343      1240  FORMAT(I2,E10.1, E10.2)
344      1290  FORMAT (7X,6H RE M ,8X,6H IM M ,8X,6H L/KYD,8X,6H RE C ,8X,6H IM C
345           1 ,5X,12H RE C BOTTOM,4X,12H IM C BOTTOM)
346       1200 CONTINUE
347            END
```

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